Public Announcement in Subset Space Logic

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1 Introduction to Subset Space Logic

In the early 90s, Moss and Parikh presented a bimodal logic called subset space logic to formalize the reasoning about sets and points. One of the modal operators was intended to quantify over the sets (□) whereas the other modal operator was intended to quantify in the sets (K). The sets in Moss and Parikh’s structure are called observation or measurement sets. The underlying motivation for the introduction of these two modalities is to be able to speak about the notion of closeness. The key idea of Moss and Parikh’s approach to the closeness can be formulated as follows: “In order to get close, one needs to spend some effort.”

1.1 Subset Space Models

The language of subset space logic $L_S$ has a countable set $P$ of proposition letters, a truth constant $\top$, the usual Boolean operators $\neg$ and $\land$, and two modal operators $K$ and $\Box$. The formulae of $L_S$ are obtained from atomic propositions by closing under $\neg$, $\land$, $K$ and $\Box$.

A Subset frame is a pair $S = \langle S, \sigma \rangle$ where $S$ is a set of points and $\sigma$ is a set of subsets of $S$. However, note that $\sigma$ is not necessarily a topology. The elements of $\sigma$ are called observations.

Definition 1.1 (Subset Space Models). A subset space model is a triple $S = \langle S, \sigma, v \rangle$ where $\langle S, \sigma \rangle$ is a subset frame, $v : P \rightarrow \wp(S)$ is a valuation function for the countable set of propositional variables $P$.

We can now define the semantics of subset spaces.

Definition 1.2 (Semantics of Subset Spaces). For $s \in S$ and $s \in U \in \sigma$, we define the satisfaction relation $|S|$ on $(S \times \sigma) \times L$ by induction on the length of the formulae. We drop the subscript $S$ when the space we are in is obvious.

\[
\begin{align*}
  s, U \models p & \quad \text{if and only if} \quad s \in v(p) \\
  s, U \models \varphi \land \psi & \quad \text{if and only if} \quad s, U \models \varphi \quad \text{and} \quad s, U \models \psi \\
  s, U \models \neg \varphi & \quad \text{if and only if} \quad s, U \not\models \varphi \\
  s, U \models K\varphi & \quad \text{if and only if} \quad t, U \models \varphi \quad \text{for all} \quad t \in U \\
  s, U \models \Box\varphi & \quad \text{if and only if} \quad s, V \models \varphi \quad \text{for all} \quad V \in \sigma \\
    & \quad \text{such that} \quad s \in V \subseteq U
\end{align*}
\]

We call $\Box$ shrinking operator and $K$ knowledge operator. The duals of $\Box$ and $K$ are $\Diamond$ and $L$ respectively and defined as follows $L\varphi \equiv \neg K
eg \varphi$ and $\Box \neg \varphi \equiv \neg \Diamond \neg \varphi$. Consequently, their semantics are defined as follows.

\[
\begin{align*}
  s, U \models L\varphi & \quad \text{if and only if} \quad t, U \models \varphi \quad \text{for some} \quad t \in U \\
  s, U \models \Diamond \varphi & \quad \text{if and only if} \quad s, V \models \varphi \quad \text{for some} \quad V \in \sigma, \\
    & \quad \text{such that} \quad s \in V \subseteq U.
\end{align*}
\]
$(s, U)$ is called a neighborhood situation if $U$ is a neighborhood of $s$, i.e. if $s \in U \in \sigma$. If at $(s, U)$ we know $\varphi$, this then means that we can move from the given reference point $s$ to any other point $t$ in the given neighborhood situation $(s, U)$. Likewise, by the shrinking modality, we shrink the neighborhood around the given point. However, neither by the knowledge nor by the shrinking modality we can leave the initial neighborhood $U$.

It is also worthwhile to remark that, subset space logic has two disjoint sets of formulae with regards to the extensions of the formulae. Considering the semantics of the subset spaces, it is not difficult to see that the truth of propositional variables and Boolean formulae do not depend on the neighborhood situations but rather depend on the points. Therefore, the extensions of these formulae are simply the points of the space. However, the extensions of the modal formulae are the neighborhood situations. Let $\mathcal{L}_0$ be the propositional language generated by the set of propositional letters $P$. Then, for the subset space frame $\mathcal{S} = \langle S, \sigma \rangle$, if $\varphi \in \mathcal{L}_0$, then we have $(\varphi)^S \subseteq S$ whereas if $\varphi \in \mathcal{L}_S - \mathcal{L}_0$, we then have $(\varphi)^S \subseteq S \times \sigma$.

1.2 Axioms

Let us now give the axioms for subset space logic. The axioms simply reflects the fact that the $K$ modality is $S5$-like whereas the $\Box$ modality is $S4$-like. Moreover, we need additional axioms to state the interaction between those two modalities. The basic axioms of subset space logic are given as follows.

1. All the substitutional instances of the tautologies of the classical propositional logic
2. $(A \rightarrow \Box A) \land (\neg A \rightarrow \Box \neg A)$ for atomic sentence $A$
3. $K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$
4. $K\varphi \rightarrow (\varphi \land KK\varphi)$
5. $L\varphi \rightarrow KL\varphi$
6. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$
7. $\Box \varphi \rightarrow (\varphi \land \Box \Box \varphi)$
8. $K\Box \varphi \rightarrow \Box K\varphi$

The rules of inferences of subset space logic are given as follows.

Modus ponens $\varphi \rightarrow \psi, \varphi \therefore \psi$

$K$-Necessitation $\varphi \therefore K\varphi$

$\Box$-Necessitation $\varphi \therefore \Box \varphi$
The last axiom is called the cross axiom and establishes the connection between shrinking and knowledge modalities. Let us see that the cross axiom in fact holds. Let us consider its dual $\Box L\varphi \to L\Box \varphi$. Assume that $s,U \models \Box L\varphi$. So, there is a subset $V \subseteq U$ such that $s,V \models \Box \varphi$. Then, again by definition there exists some $t$ in $V$ such that $t,V \models \varphi$. As $V \subseteq U$, we derive that $t,U \models \Box \varphi$. Recall that $t \in V$, hence $t \in U$. Thus, $s,U \models L\Box \varphi$.

The axiom $L\varphi \to KL\varphi$ is also worth considering. Consider its contrapositive $LK\varphi \to K\varphi$. Assume that $s,U \models LK\varphi$. Then, for some $t \in U$, we have $t,U \models K\varphi$. So, for each $u \in U$ we conclude $u,U \models \varphi$. However, this means $s,U \models K\varphi$.

The axiom $(A \to \Box A) \land (\neg A \to \Box \neg A)$ for an atomic sentence $A$ is called the axiom of atomic permanence. Intuitively, it says that, the truth of an atomic sentence $A$ at a point $s$ is independent from its neighborhood. In other words, whichever neighborhood of $s$ we consider, the truth of atomic sentences remains intact.

Then, the soundness of the axioms can easily be shown by an induction on the length of derivations.

**Theorem 1.1 (Soundness of Subset Space Axioms).** The axioms of subset space are sound.

**Proof.** Straightforward.

On the other hand, the completeness and decidability of subset spaces are not trivial. We refer the reader to the original papers for the sophisticated treatment of these results. For now, we will only state them.

**Theorem 1.2 (Completeness).** The basic axioms are strongly complete for subset space models.

**Theorem 1.3 (Decidability).** The subset space logic is decidable.

We will call the system whose axioms are the basic axioms given in Section 1.2 together with WDA and UA topologic. The underlying idea behind topologic is to create a system which is “strong enough to support elementary topological reasoning” (Moss et al., 2007).

The axioms of topologic hold in all topological spaces, but there is more. Georgatos showed the following (Georgatos, 1997).

**Theorem 1.4.** Topologic axioms are complete for topological spaces, and moreover for complete lattice spaces. Furthermore, any sentence satisfiable in any topological space is also satisfiable in a finite topological space.

The proof uses some sophisticated mathematical techniques (such as splittings) and hence is omitted here.

### 1.3 Overlap Modality

Recall that the semantics of subset space logic does not let us leave the particular neighborhood occupied. We can either move to a point in the neighborhood
or else shrink the neighborhood keeping the reference point intact. In (Heinemann, 2006), Heinemann introduced the overlap operator $O$ with the intention of leaving the current neighborhood and moving to another neighborhood of the same point. As he pointed out, the overlap operator was designed to enable us to quantify “not only downwards, but also diagonally” among the set of observations (Heinemann, 2006). Therefore, overlap operators will also include shrinking cases (but not the other way around, obviously). Hence, we can say that shrinking is a special case of overlapping.

The semantics of the overlap operator $O$ is given as follows:

$$s, U \models O \varphi \text{ iff } \forall U' \in \sigma: (s \in U' \rightarrow s, U' \models \varphi)$$

The dual of $O$ is denoted by $P$, and defined in the usual way. As we already pointed out, $\Box$ is a special case of $O$. This observation is captured by the following statement: $\models O \varphi \rightarrow \Box \varphi$.

The complete axiomatization of the topologic with the additional overlap operator was given by Heinemann in (Heinemann, 2006). The additional axioms for the overlap operator are as follows:

1. $O(\varphi \rightarrow \psi) \rightarrow (O \varphi \rightarrow O \psi)$
2. $(A \rightarrow OA) \land (PA \rightarrow A)$ for atomic $A$
3. $O \varphi \rightarrow OO \varphi$
4. $\varphi \rightarrow OP \varphi$
5. $O \varphi \rightarrow \Box \varphi$

The proof theory of subset space logic extended with the overlap operator uses Modus Ponens, $K$-necessitation, $\Box$-necessitation and $O$-necessitation in its proof theory. Moreover, together with the axioms of topologic, they completely axiomatize the topologic with overlap operator. The completeness and decidability proofs of the extended subset space logic can be found in (Heinemann, 2006). The important observation that Heinemann notes is that the overlap operator dismisses the set component of the neighborhood situation. Furthermore, it is not definable in the language of topologic.

Completeness of the extension of subset space logic with overlap operator is rather similar to that of basic subset space logic. One only needs to introduce the necessary properties which comply with the overlap operator to the completeness proof. We refer the reader to the aforementioned article for rather straightforward construction (Heinemann, 2006).

## 2 Validity Preserving Operations

Some basic validity preserving operations can easily be defined in the context of subset space logic. Let us start with the simplest one.
2.1 Disjoint Unions

Disjoint unions are perhaps the most intuitive way to obtain larger structures. We start by defining the disjoint unions for the general case as follows.

**Definition 2.1** (Disjoint Unions). Two subset space models are disjoint if their domain contains no common element. For disjoint subset space models $S_i = \langle S_i, \sigma_i, v_i \rangle$, for $i \in I$ their disjoint union is the structure $S = \bigcup_{i \in I} S_i = \langle S, \sigma, v \rangle$ where $S = \bigcup_{i \in I} S_i$, $\sigma = \bigcup_{i \in I} \sigma_i$ and $v(p) = \bigcup_{i \in I} v_i(p)$.

If the subset spaces we want to put together for disjoint unions have some common elements, the easiest way to overcome this problem is to index these members in order to make them distinct.

The details for the construction of disjoint unions for basic modal language with some easy examples can be found in (Blackburn et al., 2001). The key proposition for disjoint unions is about the truth invariance of modal formulae under the construction of disjoint unions. The proof of the aforementioned proposition can also be found in (Blackburn et al., 2001). Furthermore, we can easily import this result to the subset spaces.

**Theorem 2.1** (Invariance Under Disjoint Unions for Subset Spaces). For disjoint subset space models $S_i$ for $i \in I$ and for each neighborhood situation $(s, U)$ in $S_i$, we have $s, U \models_S \varphi$ if and only if $s, U \models_{S_i} \varphi$, for each formula $\varphi$ in the language of subset space logic $L_S$.

**Proof.** The proof is by induction on the length of the formulae. The case for propositional variables and Boolean cases are easy and hence skipped.

Let us consider the case for $\varphi \equiv K \psi$. Assume $s, U \models_S K \psi$. Then for each $t \in U$, we have $t, U \models_S \psi$. However, by induction hypothesis $t, U \models_{S_i} \psi$. As $\sigma = \bigcup_{i \in I} \sigma_i$, $(t, U)$ is a neighborhood situation in $S$. So, $s, U \models_S K \psi$.

Converse direction is similar.

The case for $\varphi \equiv \Box \psi$ is also very similar. □

However, if we want to discuss whether some topological properties are definable in subset space logic, then we have to require that the underlying space should have topology, not just a collection of subsets.

Therefore we need to revise the Definition 2.1 in such a way that obtained disjoint union will be a topological space.

**Definition 2.2** (Topologic Disjoint Unions). For disjoint topologic subset space models $S_i = \langle S_i, \sigma_i, v_i \rangle$, for $i \in I$ their disjoint union is the structure $S = \bigcup_{i \in I} S_i = \langle S, \sigma, v \rangle$ where $S = \bigcup_{i \in I} S_i$, $\sigma = \{ U \in S : \forall i \in I, U \cap S_i \in \sigma_i \}$ and $v(p) = \bigcup_{i \in I} v_i(p)$.

It is then easy to check topological disjoint unions are truth preserving.

**Proposition 2.1.** Compactness and connectedness are not definable in topologic subset space logic.

**Proof.** We refer the interested reader to (Cate et al., 2006) as there is no need to reproduce the same proof here. □
2.2 Generated Subset Spaces

Disjoint unions enable us to obtain larger spaces from smaller ones. However, we can also do the reverse. In other words, we can throw away the points in such a way that the satisfiability relation will not be affected after this operation. But, what kind of points we can throw away? Recall that, in subset spaces, we interpret the formulae at the neighborhood situations \((s, U)\) where \(s \in U \in \sigma\). Therefore, we claim that we can throw away the points which are not in any of the observation sets in \(\sigma\). The reason for that is the fact that we cannot evaluate the formulae at these points as there are no observation sets attached to them. Intuitively, they are the points about which we did not make any observation, hence there is no harm to throw them away.

**Proposition 2.2.** Let \(S = \langle S, \sigma, v \rangle\) be a subset space model. Let \(S' = S - \{s : s \notin \bigcup \sigma\}\) and \(v'(p) = v(p) \cap S'\). Then \(S' = \langle S', \sigma', v' \rangle\) and \(S = \langle S, \sigma, v \rangle\) satisfy the same formulae.

**Proof.** It is easy to show that \(s, U \models_S \varphi\) implies \(s, U \models_S \varphi\) as \(S' \subseteq S\).

For the converse direction, assume \(s, U \models_S \varphi\). As \(s \in U \in \sigma\), we observe \(s \in \bigcup \sigma\). Hence \(s \in S'\). The proof goes by induction on the complexity of \(\varphi\). The propositional and Boolean cases are easy and hence skipped. However, the idea for \(\varphi \equiv K\) and \(\varphi \equiv \Box \psi\) is very similar. We will only show the case for \(\varphi \equiv K\), and leave the other case to the reader.

Assume, \(s, U \models_S K\psi\). Then for each \(t \in U\), we have \(t, U \models_S \psi\). As \(t \in U \in \sigma\), we observe \(t \in S'\) for each \(t \in U\). Then by the induction hypothesis, \(t, U \models_{S'} \psi\). As \(s \in S'\) as well, we conclude \(s, U \models_S K\psi\). □

We can go further. We can consider a situation in which we are only interested in the observations and knowledge around one specific neighborhood situation \((s, U)\). We can then throw away the observation sets which have an empty intersection with the designated observation set \(U\). We will call this construction generated subset spaces.

**Definition 2.3 (Generated Subset Spaces).** Let \(S = \langle S, \sigma, v \rangle\) be a subset space model. Let \((s, U)\) be the designated neighborhood situation. Then we obtain the generated subset space \(S' = \langle S', \sigma', v' \rangle\) of \(S\) as follows.

- \(\sigma' := \sigma - \{V \in \sigma : V \notin U\}\)
- \(S' := S - \bigcup \sigma'\)
- \(v'(p) := v(p) \cap S'\) for each propositional letter \(p\).

**Proposition 2.3 (Invariance Under Generated Subset Spaces).** For each \(s \in S'\), we have \(s, U \models_S \varphi\) if and only if \(s, U \models_{S'} \varphi\).

**Proof.** We leave the easy proof to the reader. □
2.3 Bounded Morphism

The third validity preserving operation we will import from the basic modal language is bounded morphism. However, in the context of subset spaces, we will diverge a bit from the familiar Kripkean notion of morphisms. We will here call a function topologic-continuous function if the inverse image of an observation set is again an observation set. Likewise we will call a function topologic-open function if the image of an observation set is still an observation set. Now, we will adopt the definition of bounded morphisms from (Cate et al., 2006).

Definition 2.4 (Bounded Morphism for Subset Space Logic). Let $S = \langle S, \sigma, v \rangle$ and $S' = \langle S', \sigma', v' \rangle$ be two subset spaces. Let $f$ be an topologic-open and topologic-continuous map from $S$ to $S'$. Then $s, U \models_S \varphi$ if and only if $s', U' \models_{S'} \varphi$ for each formula $\varphi$. We define $v(p) := f^{-1}(v'(p))$.

Again, some basic facts and results about the bounded morphisms for the basic modal language can be found in (Blackburn et al., 2001). Some further observations about bounded morphism in the context of topological interpretation of modal logic can be found in (Cate et al., 2006).

Henceforth, we can claim the following invariance result.

Theorem 2.2 (Invariance Under Bounded Morphism). Let $f$ be a bounded morphism from $S = \langle S, \sigma, v \rangle$ onto $S' = \langle S', \sigma', v' \rangle$. Then $s, U \models_S \varphi$ if and only if $fs, fU \models_{S'} \varphi$ for each formula $\varphi$ in the language of subset space logic $L_S$.

Proof. The proof is by induction on the complexity of $\varphi$. The propositional case and Boolean cases are easy and hence skipped.

Let $\varphi \equiv K\psi$. Assume $s, U \models_S K\psi$. Then for each $t \in U$, we have $t, U \models_S \psi$. By induction hypothesis, we have $ft, fU \models_{S'} \psi$. As $f$ is open $fU \in \sigma'$. As $fs \in fU$, we conclude $fs, fU \models_{S'} K\psi$. The converse direction is very similar. We only need to replace $f$ by $f^{-1}$ and use the continuity of $f$.

Let $\varphi \equiv \Box\psi$. Assume $s, U \models_S \Box\psi$. Then there exists $V$ with $s \in V \subseteq U$ such that $s, V \models_S \psi$. Then by induction hypothesis we have $fs, fV \models_{S'} \psi$. As $f$ is open $fV$ is in $\sigma'$ and since $V \subseteq U$, we then trivially have $fV \subseteq fU$. Hence, $fs, fU \models_{S'} \Box\psi$. The converse direction is also very similar to that of the previous case.

Hence the induction is complete. $lacksquare$

2.4 Bisimulation

Bisimulation is an equivalence relation which is defined, in the most general setting, between state transition systems. As Kripke models can also be seen as labelled state transition systems, bisimulations can also be defined in the context of Kripke semantics. In this part, we will simply adopt the ideas from basic modal logic to define bisimulations in topologic. In the context of subset space logic, the introduction of the notion of bisimulations does not provide anything new, but instead points out the similarities of subset space logics with labelled transition systems of Kripke models.
Definition 2.5 (Bisimulation). Let $S = (S, \sigma, u)$ and $T = (T, \tau, v)$ be two subset spaces. A topologic bisimulation is a non-empty relation $\rightsquigarrow$ for neighborhood situations in $(S \times \sigma) \times (T \times \tau)$ such that if $(s, U) \rightsquigarrow (t, V)$, then we have:

1. **Base Condition**

   $s \in u(p)$ if and only if $t \in v(p)$ for each propositional variable $p$

2. **Back Conditions**

   (a) $\forall t' \in V$ there exists $s' \in U$ with $(s', U) \rightsquigarrow (t', V)$.

   (b) $\forall V' \subseteq V$ such that $t \in V'$, there is $U' \subseteq U$ with $s \in U'$ such that $(s, U') \rightsquigarrow (t, V')$

3. **Forth Conditions**

   (a) $\forall s' \in U$ there exists $t' \in V$ with $(s', U) \rightsquigarrow (t', V)$.

   (b) $\forall U' \subseteq U$ such that $s \in U'$, there is $V' \subseteq V$ such that $(s, U') \rightsquigarrow (t, V')$.

The immediate result of this definition is the Bisimulation Invariance Theorem for Subset Spaces.

Theorem 2.3 (Bisimulation Invariance for Subset Spaces). If $(s, U) \rightsquigarrow (t, V)$ then they satisfy the same formulae.

Proof. The proof is by the induction on the length of formulae. The Boolean cases are obvious. Let us start with the $\diamond$ operator. Assume $(s, U) \rightsquigarrow (t, V)$ and further $s, U \models \diamond \varphi$. So for some $U' \subseteq U$ with $s \in U'$, we have $s, U' \not\models \varphi$. Then, by the back condition (b), we have $V' \subseteq V$ with $(s, U') \rightsquigarrow (t, V')$. By the induction step, we conclude $t, V' \models \varphi$. As $V' \subseteq V$, the last statement means $t, V \models \diamond \varphi$. This concludes the one direction of the proof. The other direction is similar.

For $L \varphi$ case, assume that $s, U \models L \varphi$ and $(s, U) \rightsquigarrow (t, V)$. Then by definition, there is some $s' \in U$ such that $s', U \models \varphi$. As $(s, U)$ and $(t, V)$ are bisimilar, by back condition (a), there exists $t' \in V$ such that $(s', U) \rightsquigarrow (t', V)$. By the induction step, we conclude $t', V \models \varphi$. Hence, as $t' \in V$ we see $t, V \models L \varphi$. This concludes the one direction of the proof. The other direction is similar.

Hence, bisimular neighborhood situations satisfy the same formulae. ■

The very first difference between the topologic bisimulations and the topo-bisimulations (See Definition ??) is the fact that the topological bisimulation is defined for the neighborhood situations whereas the topo-bisimulations are defined for the states. Second, the topologic bisimulations are more general than the topo-bisimulations for obvious reasons.

However, the converse of this statement is not necessarily true. If some restrictions are applied to the topologic spaces, converse of the Theorem 2.3 can be obtained.
Before presenting the converse of the Theorem 2.3, let us introduce the following notation. \((s, U) \rightsquigarrow (t, V)\) will denote that \((s, U)\) and \((t, V)\) satisfy the same formulae.

**Theorem 2.4.** Let \(S = \langle S, \sigma, u \rangle\) and \(T = \langle T, \tau, v \rangle\) be two finite subset space. Then for each neighborhood situations \((s, U)\) in \(S \times \sigma\) and \((t, V)\) in \(T \times \tau\); we have \((s, U) \rightleftharpoons (t, V)\) if and only if \((s, U) \rightsquigarrow (t, V)\).

**Proof.** Sufficiency condition has been proven in Theorem 2.3. For the necessity condition, the idea is to show that \(rightleftharpoons\) is itself a bisimulation. So, assume \((s, U) \rightleftharpoons (t, V)\). Then, the base condition of the bisimulation is satisfied immediately. For the forth condition, let us assume further that \(t' \in V\). We will try to get a contradiction by assuming that there is no \(s' \in U\) such that \((s', U) \rightleftharpoons (t', V)\). Now, for each \(t_i'\) for \((i \leq n, \text{ for some } n \text{ as } V \text{ is finite})\) there exists some formula \(\psi_i\) such that \(s', U \models \psi_i\) whereas \(t_i', V \not\models \psi_i\). Hence, \(s, U \models \bigwedge_{i \leq n} \psi_i\) whereas \(t, V \not\models \bigwedge_{i \leq n} \psi_i\). However, we assumed that \((s, U) \rightleftharpoons (t, V)\), so we get a contradiction. Then we see that \(rightleftharpoons\) satisfies the forth condition (a). The back condition (a) is very similar.

Next, let us assume again \((s, U) \rightleftharpoons (t, V)\). To get a contradiction, now we will assume that for \(V' \subseteq V\) with \(t \in V'\), there is no corresponding \(U' \subseteq U\) with \(s \in U'\) such that \((s, U') \rightleftharpoons (t, V')\). Then for each \(V'_i \subseteq V\) with \(t \in V'\) and \(U' \subseteq U\) with \(s \in U'\), there exist some formula \(\psi_i\) (for \(i \leq n\) for some finite \(n\)) such that \(s, U' \models \psi_i\) whereas \(t, V'_i \not\models \psi_i\). But, as \(V\) is finite, then so is the number of \(V's\). Then in a similar fashion, we see \(s, U \models \bigwedge_{i \leq n} \diamond \psi_i\) whereas \(t, V \not\models \bigwedge_{i \leq n} \diamond \psi_i\). However, we assumed that \((s, U) \rightleftharpoons (t, V)\), so we get a contradiction. Then we see that \(rightleftharpoons\) satisfies the forth condition (b). The back condition (b) is very similar.

Hence the proof is complete. ■

In this proof, we simply incorporated the observations from basic modal logic. The proof of Hennesy - Milner Theorem in (Blackburn et al., 2001) establishes the main argument of the proof.

In a similar manner, it is not difficult to define bisimulations for cross axiom models. The definition is as follows.

**Definition 2.6** (Bisimulation on Cross Axiom Models). Let \(M_1 = \langle M_1, L_1, O_1, V_1 \rangle\) and \(M_2 = \langle M_2, L_2, O_2, V_2 \rangle\) be two cross axiom models. A cross axiom bisimulation is a non empty relation \(\rightleftharpoons \subseteq M_1 \times M_2\) such that if \(M_1, w_1 \rightleftharpoons M_2, w_2\) then we have:

1. **Base Condition**
   
   \(w_1\) and \(w_2\) satisfy the same propositional variables.

2. **Back Conditions**
   
   **(a)** If \(w_2 \xrightarrow{L_2} v_2\), then there exists \(v_1 \in M_1\) with \(v_1 \rightleftharpoons v_2\) and \(w_1 \xrightarrow{L_1} v_1\)
   
   **(b)** If \(w_2 \xrightarrow{O_2} v_2\), then there exists \(v_1 \in M_1\) with \(v_1 \rightleftharpoons v_2\) and \(w_1 \xrightarrow{O_1} v_1\)
3. Forth Conditions

(a) If \( w_1 \xrightarrow{L_1} v_1 \), then there exists \( v_2 \in M_2 \) with \( v_1 \xleftrightarrow{L_2} v_2 \).

(b) If \( w_1 \xrightarrow{\diamond_1} v_1 \), then there exists \( v_2 \in M_2 \) with \( v_1 \xleftrightarrow{\diamond_2} v_2 \).

The usual consequence of the definition is the cross axiom bisimulation invariance between cross axiom models.

**Theorem 2.5** (Cross Bisimulation Invariance). If \( M, w \xleftrightarrow{} M', w' \) then \( w \) and \( w' \) satisfy the same formulae.

**Proof.** The proof is a straightforward induction on the length of the formulae. We then leave it to the reader. ■

2.5 Topologic Games

We can play games in subset spaces. We will here consider two kinds of games. **Topologic Evaluation Games** will evaluate whether a given formula \( \varphi \) holds in a given neighborhood \( (s, U) \). The **Topological Bisimulation Games**, on the other hand, will evaluate whether two topologic spaces are bisimilar or not.

The players will be Eloise (denoted by \( \exists \)) and Abelard (denoted by \( \forall \)).

2.5.1 Topologic Evaluation Games

We will calculate whether a given formula \( \varphi \) holds in a given neighborhood \( (s, U) \) by topologic evaluation games. The positions in the game will be of the form \( (\varphi, (s, U)) \) where \( \varphi \) is a well-formed formula in the language of basic topologic and \( (s, U) \) is a neighborhood situation. We will work with formulae in the positive normal form. Recall that a formula \( \varphi \) is in positive normal form if \( \varphi \) has no negation symbol, or \( \varphi \equiv \neg \psi \) where \( \psi \) has no negation symbol. Observe that, \( \neg \) symbol changes the roles of \( \forall \) and \( \exists \).

Hence the topologic evaluation game \( E(\varphi, S) \) for topologic space \( S \) is a board game with players \( \exists \) and \( \forall \) moving a token around the positions of the form \( (\psi, (s, U)) \) where \( \psi \) is a subformula of \( \varphi \) and \( (s, U) \) is a given neighborhood situation. The rules of the game is given below.

<table>
<thead>
<tr>
<th>Position</th>
<th>Player</th>
<th>Admissible Moves</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\bot, (s, U)) )</td>
<td>( \exists )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( (\top, (s, U)) )</td>
<td>( \forall )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( (p, (s, U)) ) with ( s \in v(p) )</td>
<td>( \forall )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( (p, (s, U)) ) with ( s \notin v(p) )</td>
<td>( \exists )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( (\psi_1 \land \psi_2, (s, U)) )</td>
<td>( \forall )</td>
<td>( { (\psi_1, (s, U)), (\psi_2, (s, U)) } )</td>
</tr>
<tr>
<td>( (\psi_1 \lor \psi_2, (s, U)) )</td>
<td>( \exists )</td>
<td>( { (\psi_1, (s, U)), (\psi_2, (s, U)) } )</td>
</tr>
<tr>
<td>( (L\psi, (s, U)) )</td>
<td>( \exists )</td>
<td>( { (\psi, (t, U)) : t \in U } )</td>
</tr>
<tr>
<td>( (K\psi, (s, U)) )</td>
<td>( \forall )</td>
<td>( { (\psi, (t, U)) : t \in U } )</td>
</tr>
<tr>
<td>( (\lozenge\psi, (s, U)) )</td>
<td>( \exists )</td>
<td>( { (\psi, (s, V)) : s \in V \subseteq U } )</td>
</tr>
<tr>
<td>( (\square\psi, (s, U)) )</td>
<td>( \forall )</td>
<td>( { (\psi, (s, V)) : s \in V \subseteq U } )</td>
</tr>
</tbody>
</table>
Negations change the roles of $\forall$ and $\exists$ in the game. Observe that, since the language of subset spaces is an extension of first order logic, evaluation games will be finite.

Winning conditions can be formulated as follows: $\exists$ wins if $\forall$ gets stuck, and dually, $\forall$ wins if $\exists$ gets stuck. As a matter of notations, we will denote the winning positions for $\exists$ by $Win_\exists(\mathcal{E}(\varphi, (s, U)))$. However, one can feel the basic intuition behind the evaluation games by considering the winning positions. We will observe in the next theorem that, if a position lies $(\varphi, (s, U))$ in the winning positions for $\exists$, then it is equivalent to say $s, U \models \varphi$.

**Theorem 2.6** (Adequacy Theorem for Topologic Evaluation Games).

$$(\varphi, (s, U)) \in Win_\exists(\mathcal{E}(\varphi, (s, U))) \text{ if and only if } s, U \models \varphi.$$

**Proof.** Proof goes by induction on the length of the formula. The case for propositional variables and Boolean operators are easy and hence skipped. The case for negation is also easy. As we work with positive normal formula, if the given formula is a negation, then the roles are swapped.

Let us consider the case for $\varphi \equiv L\psi$. Assume we are in the position $(L\psi, (s, U))$. Then it is $\exists$'s turn. She will pick a point $t$ in $U$ and move to $(\psi, (t, U))$. As $(L\psi, (s, U))$ is a winning position for $\exists$, then so is $(\psi, (t, U))$. Then by induction hypothesis we see $t, U \models \psi$. But then, as $s \in U$ as well, we obtain $s, U \models L\psi$. For the converse direction, assume we have $s, U \models L\psi$. To get a contradiction assume further that $(L\psi, (s, U))$ is not a winning situation for $\exists$ which means that after each move out of $(s, U)$ under the formula $L\psi$, $\exists$ will lose eventually. Now, as $s, U \models L\psi$, for some $t \in U$ we then have $t, U \models \psi$. By induction hypothesis, we conclude, $(\psi, (t, U)) \in Win_\exists(\mathcal{E}(\psi, (t, U)))$. However, we observed that $(L\psi, (s, U))$ is not a winning situation for $\exists$, then, $(\psi, (t, U))$ cannot be a winning situation for $\exists$ neither. Contradiction shows that, $(L\psi, (s, U))$ is a winning position for $\exists$.

Let us now consider the case $\varphi \equiv K\psi$. Assume we are in the position $(K\psi, (s, U))$ which is a winning condition for $\exists$. However, now it is $\forall$'s turn. He will pick any point $t$ in $U$ and move to $(\psi, (t, U))$. But, for each $t \in U$, the position $(\psi, (t, U))$ is still a winning position for $\exists$. Hence, $t, U \models \psi$. But, this statement holds for each $t \in U$, and hence $s, U \models K\psi$. For the converse direction assume $s, U \models K\psi$. To get a contradiction assume further that $(K\psi, (s, U))$ is not a winning situation for $\exists$ which means that after each move of $\forall$ out of $(s, U)$ with the formula $K\psi$, $\exists$ will lose eventually. Now, as $s, U \models K\psi$, for each $t \in U$ we then have $t, U \models \psi$. By induction hypothesis, we conclude, $(\psi, (t, U)) \in Win_\exists(\mathcal{E}(\psi, (t, U)))$. However, we observed that $(K\psi, (s, U))$ is not a winning situation for $\exists$, then, $(\psi, (t, U))$ cannot be a winning situation for $\exists$ neither. Contradiction shows that, $(K\psi, (s, U))$ is a winning position for $\exists$.

Let us now consider the case for $\varphi \equiv \Diamond\psi$. Assume $\Diamond\psi, (s, U)$ is a winning position for $\exists$. So, when we are in the position $\Diamond\psi, (s, U)$, $\exists$ will pick a subset $V$ of $U$ with $s \in V$ and move to the position $\psi, (s, V)$. By induction hypothesis, we conclude $s, V \models \psi$. But now, it is easy to see that $s, U \models \Diamond\psi$. For the other direction, let us assume $s, U \models \Diamond\psi$ holds. To get a contradiction, assume
$(\Diamond \psi, (s, U))$ is not a winning position for $\exists$. From, $s, U \models \Diamond \psi$ we conclude $s, V \models \psi$ for some $s, \in V \subseteq U$. By induction hypothesis we see $(\psi, (s, V))$ is a winning position for $\exists$ in the game $E(\psi, (s, V))$. However, we assumed that $(\Diamond \psi, (s, U))$ was not a winning position, so $(\psi, (s, V))$ cannot be a winning position in the game $E(\psi, (s, V))$. The contradiction shows that, $(\Diamond \psi, (s, U))$ is a winning position for $\exists$ in the game $E(\Diamond \psi, (s, U))$.

The case for $\varphi \equiv \square \psi$ is similar.

Hence the theorem is proved.  

\medskip

2.5.2 Topologic Bisimulation Games

Topologic bisimulation games will provide an alternative semantics to approach the topologic bisimulations. In this game, $\forall$ and $\exists$ will compare neighborhood situations across the respective topologic spaces. $\exists$ wins if the given two neighborhood situations are bisimilar, $\forall$ wins otherwise.

Assume we are given $(s, U)$ and $(t, V)$. $\forall$ starts. He can either pick another point $t'$ in $V$ (or $s'$ in $U$) or pick a subset $U' \subseteq U$ such that $t \in V'$ (or a subset $U' \subseteq U$ such that $s \in U'$). If he picked another point $t'$ in $V$, then $\exists$ must find a corresponding point $s'$ in $U$ such that $(s', U) \rightsquigarrow (t', V)$. She loses immediately, if she cannot find such a point. If $\forall$ picked a subset $U' \subseteq U$ with $t \in V'$, then $\exists$ must find a corresponding subset $U' \subseteq U$ with $s \in U'$ such that $(s, U') \rightsquigarrow (t, V')$. She loses immediately if she cannot find such a subset.

A topologic bisimulation game of length $n$, then can be defined as a game which can distinguish formulas of depth at most $n$. It is then easy to observe that, $\exists$ has a winning strategy in the bisimulation game of length $n$ for $(s, U)$ and $(t, V)$ if and only if these two neighborhood situations are actually bisimilar for formulas of depth at most $n$.

It is easy to see that when $\exists$ has a winning strategy in the bisimulation game, then the neighborhood situations are bisimilar. The proof goes by induction on the depth of the formulae and the application of straight forward ideas.

To see the converse, assume we have a bisimilar neighborhood situations $(s, U)$ and $(t, V)$. Then by following the definition of bisimulation, we will form a winning strategy for $\exists$. $\forall$ starts the game. He can pick a point $t'$ in $V$. But as $(s, U) \rightsquigarrow (t, V)$, we can find a corresponding point $s' \in U$ such that $(s', U) \rightsquigarrow (t', V)$. If he picked a point $s'' \in U$, then by similar arguments, we can find $t'' \in V$ such that $(s'', U) \rightsquigarrow (t'', V)$. Therefore, we can add these points to the winning strategy of $\exists$. On the other hand, $\forall$ can pick the subsets $V' \subseteq V$ where $t \in V'$ or $U' \subseteq U$ where $s \in U'$. But in any case, by following the above argumentation, we again come up with a corresponding subsets which maintain the bisimulation as $(s, U) \rightsquigarrow (t, V)$. Hence, we established a winning strategy for $\exists$ by just following the bisimulation relation $\rightsquigarrow$.

Hence we proved the Adequacy Theorem for Bisimulation Games.

**Theorem 2.7 (Adequacy Theorem for Topologic Bisimulation Games).** $(s, U) \rightsquigarrow_n (t, V)$ if and only if $\exists$ has a winning strategy in the topologic bisimulation game of length $n$.

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Proof. Given above the theorem.

2.5.3 Topologic Games in Extended Languages

Topologic games can easily be defined in extended languages. Therefore, in order to see that, we will show on example - subset space logic with overlap operator.

Heinamann’s overlap operator $O$ enables us to change the current neighborhood. Therefore, we can have much more freedom to move around the subset space. Then, the rules for this game is as follows. The rules of the game is given below.

<table>
<thead>
<tr>
<th>Position</th>
<th>Player</th>
<th>Admissible Moves</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\perp, (s, U))$</td>
<td>$\exists$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$(\top, (s, U))$</td>
<td>$\forall$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$(p, (s, U))$ with $s \in v(p)$</td>
<td>$\forall$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$(p, (s, U))$ with $s \notin v(p)$</td>
<td>$\exists$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$(\psi_1 \land \psi_2, (s, U))$</td>
<td>$\forall$</td>
<td>${(\psi_1, (s, U)), (\psi_2, (s, U))}$</td>
</tr>
<tr>
<td>$(\psi_1 \lor \psi_2, (s, U))$</td>
<td>$\exists$</td>
<td>${(\psi_1, (s, U)), (\psi_2, (s, U))}$</td>
</tr>
<tr>
<td>$(L\psi, (s, U))$</td>
<td>$\forall$</td>
<td>${(\psi, (t, U)) : t \in U}$</td>
</tr>
<tr>
<td>$(K\psi, (s, U))$</td>
<td>$\exists$</td>
<td>${(\psi, (t, U)) : t \in U}$</td>
</tr>
<tr>
<td>$(\Diamond\psi, (s, U))$</td>
<td>$\exists$</td>
<td>${(\psi, (s, V)) : s \in V \subseteq U}$</td>
</tr>
<tr>
<td>$(\Box\psi, (s, U))$</td>
<td>$\forall$</td>
<td>${(\psi, (s, V)) : s \in V \subseteq U}$</td>
</tr>
<tr>
<td>$(P\psi, (s, U))$</td>
<td>$\exists$</td>
<td>${(\psi, (s, U')) : s \in U'}$</td>
</tr>
<tr>
<td>$(O\psi, (s, U))$</td>
<td>$\forall$</td>
<td>${(\psi, (s, U')) : s \in U'}$</td>
</tr>
</tbody>
</table>

The adequacy theorem for extended topologic games is then straightforward.

Theorem 2.8 (Adequacy Theorem for Extended Topologic Evaluation Games).

$(\varphi, (s, U)) \in \text{Win}_2(E(\varphi, (s, U)))$ if and only if $s, U \models \varphi$.

Proof. A simple extension of the Adequacy Theorem for Topologic Bisimulation Games - Theorem 2.7. Left as an exercise for the reader.

3 Public Announcement Logic

3.1 Introduction

Our main goal in this chapter is to give a subset space semantics for Public Announcement Logic (PAL, for short). The very first intuition for considering subset space semantics for PAL simply stems from the idea that the two modal operator of subset space logic can very well formalize the change in the knowledge after a public announcement. As we already noted, the shrinking operator has a dynamic nature.
In this chapter, starting from a familiar example of PAL, we will indicate some important motivations to formalize PAL in a geometrical (almost topological) setting. Then we will present the formal tools and develop them a bit further to get some more insights. Finally, the completeness of PAL for subset space semantics should not be a surprise. Our main reference for PAL is (Benthem et al., 2005).

This section, on the other hand, can be read as a formal introduction for the next chapter in which we will extend the discussion with further examples.

3.2 Card Showing Game

Card showing game is one of the very simple examples to illustrate the main idea of public announcements. Suppose that we are playing a simple card game with three people A, B and C and three cards p, q and r. The aim of the game is to guess the card of each player. Let the actual distribution of the cards as follows: A has p, B has q and finally C has r. Suppose further that the players have looked at their cards but kept them hidden from the other players. For the sake of notation let the ordered tuple \( \langle \alpha, \beta, \gamma \rangle \) denote the situation that the player A has the card \( \alpha \) whereas the players B and C have the cards \( \beta \) and \( \gamma \) respectively. As the order matters \( \langle \alpha, \beta, \gamma \rangle \neq \langle \gamma, \beta, \alpha \rangle \).

Therefore, in this situation, let us consider the set of observations for the agent A. As she already knows which card she has, and does not know the cards of the other people, her set of observations would be the following set \( o = \{ \langle p, q, r \rangle, \langle p, r, q \rangle \} \). Hence, by spending some effort such as cheating, bribing another player, agent A can increase her knowledge. But, the increase in the knowledge of A could also come in the form of public announcement (However, note that, in this simple game, when a public announcement of the form “Player X has the card x” is made, players other than X win the game.). Hence, if it is announced that the player B has card q, then the set \( o \) will shrink to, say \( v = \{ \langle p, q, r \rangle \} \). Similarly, the public announcement that \( B \) has \( r \) would lead to a shrinking of \( o \) to \( w = \{ \langle p, r, q \rangle \} \). Likewise for the public announcement for player C.

Hence, we can consider the public announcements as the (external) efforts spent. What motivates us from this simple example is the fact that, we can make the shrinking procedure explicit by a public announcement. Then why not identify this procedure with a mapping? Recall that we start with the set \( o \) and by throwing the refutative elements away from the set, we obtained the subsets \( v \subseteq o \) or \( w \subseteq o \) of \( u \).

3.3 Formal Tools

Public announcement logic is typically interpreted on Kripke structures. So, before presenting the subset space semantics for PAL, let us review the Kripkean interpretation of PAL. Notationwise, the formula \( [\varphi] \psi \) is intended to mean that after the public announcement of \( \varphi \), \( \psi \) holds. As usual, \( K_i \) is the epistemic modality for the agent \( i \). Likewise, \( R_i \) is the epistemic accessibility relation for the
agent $i$ and $R$ stands for the set of accessibility relations. The language of PAL will be that of epistemic logic with an additional public announcement operator $\ast$ where $\ast$ can be replaced with any well formed formula.

**Definition 3.1** (Semantics of PAL). Let $\mathcal{M} = \langle W, R, V \rangle$ be a model and $i$ be an agent. For atomic propositions, negations and conjunction the definition is as usual. For modal operators, we have the following semantics:

- $\mathcal{M}, w \models K_i \varphi$ if \ and only if $\mathcal{M}, v \models \varphi$ for each $v$ such that $(w, v) \in R_i$.
- $\mathcal{M}, w \models [\varphi] \psi$ if $\mathcal{M}, w \models \varphi$ implies $\mathcal{M}|\varphi, w \models \psi$.

Here the updated model $\mathcal{M}|\varphi = \langle W', R', V' \rangle$ is defined by restricting $\mathcal{M}$ to those states where $\varphi$ holds. Define $(\varphi)^\mathcal{M} = \{v \in W : \mathcal{M}, v \models \varphi\}$. Hence, $W' = \{w \in W : w \models \varphi\}$, i.e. $W' = W \cap (\varphi)^\mathcal{M}$; $R'_i = R_i \cap (W' \times W')$ and finally $V'(p) = V(p) \cap W'$.

The proof system of public announcement logic is the proof system of multi-modal S5 epistemic logic with the following additional axioms.

- **Atoms** $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
- **Partial Functionality** $[\varphi] \neg \psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$
- **Distribution** $[\varphi](\psi \land \chi) \leftrightarrow (([\varphi]\psi \land [\varphi]\chi)$
- **Knowledge Announcement** $[\varphi]K_i \psi \leftrightarrow (\varphi \rightarrow K_i [\varphi]\psi)$

The rule of inference is called the announcement generalization and is described as follows.

- From $\vdash \psi$, derive $\vdash [\varphi]\psi$.

For the sake of simplicity, we leave the common knowledge out in the definitions. The interested reader is referred to (Benthem et al., 2005).

The key idea is that, after the public announcement $\varphi$, the states which are incompatible with $\varphi$ are discarded. In other words, the effort spent by the public announcement, makes our accessible states smaller by getting rid of refutative observations (recall that $R'_i = R_i \cap (W \times W')$ in the new model $\mathcal{M}|\varphi$). However, throughout the process, we do not change our point of view, i.e. our state. Therefore, the public announcement $\varphi$ makes sense if the current state realizes $\varphi$. Otherwise the state of the observer itself will be discarded, too. This observation manifests itself in the axioms.

### 3.4 Subset Space PAL

Following the aforementioned motivations and considerations, let us now formalize the public announcement logic in the language of the basic subset space logic. As we already pointed out, in order to eliminate some of the refutative observations, we will use the public announcements. Let us make this observation more explicit. An illustrative example can be given by following the speed measurement of the policeman case. Let us recall that the speed limit is 50 mph and policeman made an measurement that a particular car has the velocity of
51 mph. As the error range is 2 mph, we have the interval (49, 53) as our set of observation. However, a public announcement can shrink this set as well. Assume, that we have the following public announcement: “The velocity of the car is not greater than 52 mph”. Hence, the updated observation set will be (49, 52). So, we have more knowledge now as we reduced the set of possibilities significantly. Examples can be multiplied easily.

In the subset space language, as we underlined before, instead of accessibility relations, we depend on neighborhood situations. Therefore, if we want to adopt public announcement logic to the context of subset space logic, we first need to focus on the fact that the public announcements shrink the observation sets for each agent. Hence, assume we are in a subset space frame \( S = (S, \sigma) \). Then, after public announcement logic \( \phi \), we will move to another subset space frame, say \( S_{\phi} = (S|\phi, \sigma_{\phi}) \) where \( S|\phi = (\phi) \) and \( \sigma_{\phi} \) is the reduced collection of subsets after the public announcement \( \phi \). The saddle point is to construct \( \sigma_{\phi} \). As we need to get rid of the refutative states, for each observation set \( U \) in \( \sigma \), we eliminate the points which do not satisfy \( \phi \). We will disregard the empty set as no neighborhood situations can be formed with empty set. Hence \( \sigma_{\phi} = \{ U_{\phi} : U_{\phi} = U \cap (\phi) \neq \emptyset, \text{ for each } U \} \).

But then, how would the neighborhood situations be effected from the public announcement? Consider the neighborhood situation \((s,U)\) and the public announcement \(\phi\). Then the statement \(s,U \models [\phi]\psi\) will mean that after the public announcement of \(\phi\), \(\psi\) will hold in the neighborhood situation \((s,U_{\phi})\). So, first we will remove the points in \(U\) who refute \(\phi\), and then \(\psi\) will hold in the updated set \(U_{\phi}\) which was obtained from the original set \(U\). Then the corresponding semantics can be suggested as follows:

\[
s, U \models [\phi]\psi \text{ if and only if } s, U \models \phi \text{ implies } s, U_{\phi} \models \psi
\]

Before checking whether this semantics satisfies the axioms of public announcement logic, let us give the language and semantics of the topologic PAL.

The language of the topologic public announcement logic interpreted in subset spaces is given as follows:

\[
\models p \mid \bot \mid \neg \phi \mid \phi \land \psi \mid \square \phi \mid K \phi \mid [\phi]\psi
\]

The semantics for topologic PAL differs only on public announcement operator whose semantics is given as follows:

\[
s, U \models [\phi]\psi \text{ if and only if } s, U \models \phi \text{ implies } s, U_{\phi} \models \psi
\]

Now, let us consider the soundness of the following axioms of basic PAL.

\[
\begin{align*}
\text{Atoms} & \quad [\phi]p \leftrightarrow (\phi \rightarrow p) \\
\text{Partial Functionality} & \quad [\phi]\neg \psi \leftrightarrow (\phi \rightarrow \neg [\phi]\psi) \\
\text{Distribution} & \quad [\phi](\psi \land \chi) \leftrightarrow ([\phi]\psi \land [\phi]\chi) \\
\text{Knowledge Announcement} & \quad [\phi]K \psi \leftrightarrow (\phi \rightarrow K [\phi] \psi)
\end{align*}
\]

The following theorem establishes the soundness of the topologic PAL.
Theorem 3.1 (Soundness of Topologic PAL). Above axioms are sound in toponologic PAL.

Proof. As the atomic propositions do not depend on the neighborhood, the Atoms axiom is satisfied by the subset space semantics of public announcement modality. To see this, assume \( s, U \models [\varphi]p \). So, by the semantics \( s, U \models \varphi \) implies \( s, U_\varphi \models p \). So for any set \( V \) where \( s \in V \), we have \( s, V \models p \). Hence, \( s, U \models \varphi \) implies \( s, U \models p \), that is \( s, U \models \varphi \rightarrow p \). Conversely, assume \( s, U \models \varphi \rightarrow p \). So, \( s, U \models \varphi \) implies \( s \in v(p) \). As \( s, U \models \varphi \), \( s \) will lie in \( U_\varphi \), thus \( (s, U_\varphi) \) will be a neighborhood situation. Thus, \( s, U_\varphi \models p \). Then, we conclude \( s, U \models [\varphi]p \).

Partial functionality and distribution axioms are also straight forward formula manipulations and hence skipped.

The important reduction axiom is the knowledge announcement axiom. Assume, \( s, U \models [\varphi]K\psi \). Suppose further that \( s, U \models \varphi \). Then we have

\[
\begin{align*}
s, U \models [\varphi]K\psi & \iff s, U_\varphi \models K\psi \\
& \iff \text{for each } t_\varphi \in U_\varphi \text{ we have } t_\varphi, U_\varphi \models \psi \\
& \iff \text{for each } t \in U, t, U \models \varphi \text{ implies } t, U \models [\varphi]\psi \\
& \iff s, U \models K(\varphi \rightarrow [\varphi]\psi) \\
& \iff s, U \models K[\varphi]\psi
\end{align*}
\]

Hence, the above axioms are sound for the subset space semantics of public announcement logic.

Let us explain the proof further. The first step is the definition of the public announcement in the subset space semantics. Then, in the second step, we unravel the knowledge modality. However, the second step says that the neighborhood situations that satisfy \( \varphi \) will satisfy \( \psi \) after an update with \( \varphi \). Hence, for each point \( t \) in \( U \), if \( \varphi \) is true at \( (t, U) \) then after an update with \( \varphi \), \( \psi \) will be true at \( (t, U) \). However, this statement is true for each \( t \) in \( U \), so we can go back to our starting point \( s \) by knowledge modality. As \( [\varphi] \) is a partial operation, that is only applicable to the neighborhood situations which satisfy \( \varphi \), we can simplify the statement in the forth step to the one in the last step. Hence, the result follows.

However, subset space logic has an indispensable modal operator, namely the effort modality. One can wonder whether we have a reduction axiom for it as well. We start with considering the statement \( [\varphi]\Box\psi \leftrightarrow (\varphi \rightarrow \Box[\varphi]\psi) \). We will call it the reduction axiom for shrinking operator. Assume, \( s, U \models [\varphi]\Box\psi \). Suppose further that \( s, U \models \varphi \). Then we have

\[
\begin{align*}
s, U \models [\varphi]\Box\psi & \iff s, U_\varphi \models \Box\psi \\
& \iff \text{for each } V_\varphi \subseteq U_\varphi \text{ we have } s, V_\varphi \models \psi \\
& \iff \text{for each } V \subseteq U, s, V \models \varphi \text{ implies } s, V \models [\varphi]\psi \\
& \iff s, U \models \Box(\varphi \rightarrow [\varphi]\psi) \\
& \iff s, U \models [\varphi]\Box\psi
\end{align*}
\]

The first step here is again the definition of \( [\varphi] \) operator. In the second step, we unravel the effort modality. We go back to the initial model in the third
step by considering the each subset of the given neighborhood. We move to
the fourth step by quantifying over the subsets of the given neighborhood by □
modality. Hence the result follows.

In conclusion, we observed that both for K and □, we have a reduction
axiom and hence reduced the complexity of the formulae in the language of
topologic PAL step by step. Starting with a formula in the language of topologic
PAL, by following the reduction axioms we discussed, eventually we will end up
with a formula in the language $L_S$ of subset space logic. This is the key idea for
the completeness of topologic PAL.

Therefore, it is easy to see that the following axiomatize the topologic-PAL:

- **Atoms**: $\varphi p \leftrightarrow (\varphi \rightarrow p)$
- **Partial Functionality**: $\varphi \neg \psi \leftrightarrow (\varphi \rightarrow \neg [\varphi] \psi)$
- **Distribution**: $\varphi (\psi \land \chi) \leftrightarrow ([\varphi] \psi \land [\varphi] \chi)$
- **Knowledge Announcement**: $\varphi K \psi \leftrightarrow (\varphi \rightarrow K[\varphi] \psi)$
- **Shrinking Reduction**: $\varphi \square \psi \leftrightarrow (\varphi \rightarrow \square[\varphi] \psi)$

Referring to the above discussions, the completeness result for topologic PAL
follows easily.

**Theorem 3.2 (Completeness of Topologic PAL).** Topologic PAL is complete with
respect to the axiom system given above.

**Proof.** By reduction axioms we can reduce each formula in the language of topologic
PAL to a formula in the language of topologic. As topologic is strongly
complete, so is topologic PAL. ■

### 3.4.1 PAL with Overlap Operator

In this section, we will enrich the language of subset space PAL with overlap
operator. Our new language will be as follows.

$p \mid \bot \mid \neg \varphi \mid \varphi \land \psi \mid \square \varphi \mid K \varphi \mid [\varphi] \psi \mid O \varphi$

Now we observe that we can come up with a reduction axiom for overlap
operator as well.

**Theorem 3.3 (Reduction Axiom for Overlap Operator).** $[\varphi] O \varphi \leftrightarrow (\varphi \rightarrow O[\varphi] \psi)$
is sound.

**Proof.** Assume $s, U \models [\varphi] O \psi$. Suppose further that $s, U \models \varphi$. Then we have,

s, U \models [\varphi] O \psi \iff s, U \varphi \models O \psi
\iff \text{for each } V \varphi \in \sigma_{\varphi}, \text{ we have}
\quad s \in V \varphi \text{ implies } s, V \varphi \models \psi
\iff \text{for } s \in V, \text{ s, V }\models \varphi \text{ implies } s, V \models [\varphi] \psi
\iff s, U \models O(\varphi \rightarrow [\varphi] \psi)
\iff s, U \models O[\varphi] \psi

■
We can celebrate the above theorem with a completeness result.

**Theorem 3.4** (Completeness of Topologic PAL with Overlap). *Topologic public announcement logic with overlap operator is complete.*

**Proof.** Recall that the subset space logic extended with overlap operator is complete. Then, the result follows. ■

## 4 Conclusion

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**References**


