Fuzzy Does Not Lie!

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Three-valued logic,
end of the critical rationality.
Imre LAKATOS,
Proofs and Refutations.

Abstract

This paper is meant to be a brief review of the paper *The Liar Paradox and Fuzzy Logic.* In this review, it is intended to avoid technicality of proofs. However, the basic ideas of the proofs will be underlined. Original article follows afterwards.

Background and Introduction

Background

In order to analyze the present paper *The Liar Paradox and Fuzzy Logic* ([LP] for short), Peano Arithmetic (PA for short) should be recalled first.

Axioms of PA that were used in [LP] are equality (that is, $S(x,y) \rightarrow [S(x,z) \rightarrow y = z]$), functionality of successor (that is $S(\overline{k}, \overline{k+1})$ where $\overline{k}$ is constant for all numeral k), addition and multiplication. As deduction rules, we will use modus ponens, generalization and the rule of induction which is:

$$\varphi(\overline{0}), (\forall x)[\varphi(x) \rightarrow (S(x,y) \rightarrow \varphi(y))] \therefore (\forall x)\varphi(x)$$

Recall that, induction axiom scheme is not sound in many-valued case since the definition of implication might arise some problems[1].

We also know that nonstandard models of PA exists and we can obtain them by adding constants. Furthermore, it is possible to interprete the constants as the points of infinity.[2]

Moreover, as stated in [LP] following accomplishments are needed to be recalled:

[1]See [5].

[2]Any text book on axiomatic set theory will help grasp the basic ideas of these facts. Since they are off-topic, we will not go into details.

- If we keep arithmetic two-valued, but let the unary predicate $Tr(x)$ be fuzzy-valued, $\lambda \equiv \neg\lambda$ need not be contradictory as long as the truth value of λ is $1/2$ in the context of predicate Łukasiewicz logic (Ł$\forall$ for short). Unary truth predicate $Tr(x)$ will be explained in following sections. This result is due to Hájek.
- Skolem had investigated the set theory in Łukasiewicz's logic schema well before Zadeh 'invented' the fuzzy logic.
- Zadeh showed that the liar's formula must have the truth value $1/2$ in the frame of possibility theory.

Notation

As the first crucial notion, we have the truth set [0, 1] in real numbers, where 1 denotes the "absolute truth" and 0 denotes the "absolute falsity".

The initial (primitive) truth functions are $(-)$ and $\Rightarrow$ where

$$(-)x = 1 - x \qquad \text{and} \qquad x \Rightarrow y = min(1, 1 - x + y)$$

We may observe that Boolean negation and implication are the special cases of $(-)x = 1 - x$ and $x \Rightarrow y = min(1, 1 - x + y)$ respectively.

We define new truth connectives:

- $x \wedge y = min(x, y)$
- $x \& y = max(0, x + y - 1)$
- $x \vee y = max(x, y)$
- $x \veebar y = min(x + y, 1)$

Note that, the first two connectives are conjunctions whereas the latter two are disjunctions; and also it is obvious that the usual Boolean connectives are special cases of these connectives.

Truth value of φ will be denoted as $\parallel \varphi \parallel_{M,v}$ where M is a non-empty set and v is the valuation. Following the Tarskian semantics[3], they are straight-forward:

- $\parallel P(x, \ldots, c, \ldots) \parallel_{M,v} = r_p(v(x), \ldots, m_c, \ldots)$ where r_p is a fuzzy relation on M with the range $[0, 1]$ in reals and v : (object variables) $\rightarrow M$ is the valuation.
- $\parallel \varphi \rightarrow \psi \parallel_{M,v} = \parallel \varphi \parallel_{M,v} \Rightarrow \parallel \psi \parallel_{M,v}$
- $\parallel \neg\varphi \parallel_{M,v} = (-) \parallel \varphi \parallel_{M,v}$
- $\parallel (\forall x)\varphi \parallel_{M,v} = inf(\parallel \varphi \parallel_{M,v'} : v' \equiv_x v)$ where $v' \equiv_x v$ means $v(y) = v'(y)$ for all variables y except possibly x.
- $\parallel (\exists x)\varphi \parallel_{M,v} = sup(\parallel \varphi \parallel_{M,v'} : v' \equiv_x v)$

Łukasiewicz logic is obtained by adding following axioms, together with the deduction rule and generalization:

[3] See [2]

- $(\forall x)\varphi(x) \to \varphi(t)$ where t is free for x in φ.
- $(\forall x)(v \to \varphi) \to (v \to (\forall x)\varphi)$ where x is not free in v.

Axioms and Rules

Łukasiewicz's axioms were given as follows:

- $\varphi \to (\psi \to \varphi)$
- $(\varphi \to \psi) \to [(\psi \to \chi) \to (\varphi \to \chi)]$
- $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
- $[(\varphi \to \psi) \to \psi] \to [(\psi \to \varphi) \to \varphi]$

Deduction rule will be Modus Ponens, as expected.

In terms of *computability theory*, it was stated in [LP] that Ł∀, the Łukasiewicz predicate logic, is not recursively axiomatizable. Yet, weak axiomatizations exist.

Moreover, it has been shown that Ł∀ is Π_2-complete in arithmetical hierarchy, i.e. Ł∀ is a Π_2 set and, every set in Π_2 is many-one reducible to Ł∀.

Unary Truth Predicate Tr

The unary truth predicate $Tr(x)$ has the axiom schema of

$$\varphi \equiv Tr(\overline{\varphi}). \tag{1}$$

We know that the extension of PA by Tr, denoted as PATr, is inconsistent in terms of binary Boolean logic. Because we have liar's formula $\lambda \equiv Tr(\overline{\lambda})$ derivable in PATr. That is, PA $\vdash \lambda \equiv \neg Tr(\overline{\lambda})$ gives PA $\vdash \lambda \equiv \neg\lambda$. The result is due to Gödel. But, as we noted in 'Background' section, we can overcome this difficulty by some methods.

We have the theory PAŁTr as the extension of PA with the unary predicate Tr and the *Tertium non datur* ([TND] for short):

Tertium Non Datur Tertium non datur for a predicate P is the formula

$$(\forall x_1, \ldots, x_n)\{P(x_1, \ldots, x_n) \vee P(x_1, \ldots, x_n)\}$$

where P is of arity n.

It was observed that if a theory proves [TND] for some predicates; then for every valuation, we have the truth value of either 0 or 1 for every formula built from those predicates, in each model of the given theory. In other words, all the formulas built from those predicates, which prove [TND], have the binary truth values.

It is also observed that Boolean propositional logic is just Łukasiewicz propositional logic with the schema $\varphi \vee \neg\varphi$ of [TND].

Aim

The aim of the paper [LP] is to show that:

1. the standard model **N** has no many-valued extension to a model of PAŁTr.
2. in two-valued arithmetic, if we let Tr having truth values in the real interval [0, 1], we get that PAŁTr is consistent and has nonstandard models.
3. if we make Tr commute with the connectives then we get another system, namely PAŁTr$_3$, which is inconsistent.

Results

We first observe that PA and PAŁ prove exactly the very same formulae. We can show this claim by using the last argument of the section on the 'unary predicate Tr' above. So we have

$$\text{PA} \vdash \varphi \text{ if and only if PAŁ} \vdash \varphi, \text{ for all sentences } \varphi.$$

Similarly, it is easy to get a diagonal lemma for PAŁTr. That is, for every formula φ with one free variable, there is a sentence ψ such that PAŁTr $\vdash \psi \equiv \varphi(\overline{\psi})$. Diagonal lemma is a generalization of the diagonalization argument in PA, and is very similar to the fixed point theorems. Moreover, similar theorems appear in mathematics frequently. For example, consider the following example taken from calculus:

> Let $f : [0,1] \to [0,1]$ be a continuous real valued function where $f(0) = 1$ and $f(1) = 0$. Then there exists a point m in the given interval such that $f(m) = m$.

To prove it simply consider the continuous function $g(x) = f(x) - x$ and make use the some trivial theorems of single valued calculus.

However, the reviewer would have liked to learn more about the importance of diagonal lemma from the paper [LP]. Because it appears that the diagonal lemma poses an indispensable method of reasoning in logic. Therefore, it should have been stressed more explicitly and in a more detailed way in [LP].

The following striking result of the paper [LP] is that the standard model **N** cannot be expanded to a model of PAŁTr, that is PAŁTr has no standard model. This theorem was proven by *reductio ad absordum* on a model M which is the expansion of **N**. The proof simply uses the definition of $Tr(\overline{\varphi})$ and considers a sentence λ which says "I am at least a little false". This sentence was formulated in [LP] as well. So, if we take the truth value of λ as either equal to 1, or less than 1, in both ways, we end up with contradiction by using simple arithmetic and model theoretic manipulations.

But, according to [LP], nonstandard models of PAŁTr can be constructed. Unfortunately, the explicit and clear construction was not given in the paper [LP]. So, the reviewer has a strong doubt whether or not this construction also mimics the construction of nonstandard models of PA. However,

[LP] gives a brief but technical answer to the question whether or not we can have a nonstandard model of PAŁTr by standard arithmetic and a nonstandard semantic. Impossibility of having such a model was explained by a very technical theorem in [LP].

Luckily, in order to make the construction of PAŁTr$_3$ more understandable and methodologically clear[4], authors mentioned the theory PAŁTr$_2$, which is the theory PAŁTr extended by the rule of induction for arbitrary formulas.

How to prove that PAŁTr$_3$ is inconsistent? Avoiding mathematical difficulties, it suffices to underline that the proof follows the definition of Tr, i.e. the equation (1), and the property of commutativity of Tr of which was added as an extra axiom schema to get PAŁTr$_3$. The crucial idea of this proof is to use the sentence λ which says "I am at least a little false".

Discussion

There has been a continuous debate on fuzzy logic for decades. As it was indicated in [5]:

> It is not much of an exaggeration to say that (...) fuzzy logic divides the world into two: those who advocate it with missionary zeal; and those who think that it is logical pornography. (It is intriguing and exciting at first sight; but quickly seen as shallow and just a little disgusting.)

However, as Pelletier underlined in [5], Hájek mainly considers fuzzy logic as an infinitely-many valued logic. Also, Hájek claims in [2] that fuzzy logic "may be classified as philosophical logic". So, we deduce that, the authors (at least Hájek) considers fuzziness as an indispensible tool for logic. But, we refrain ourselves to conclude whether or not Hájek 'advocate it with missionary zeal'. Actually, paper speaks for itself.

It is also easy to find some more examples to help get the idea about the relations, for example, between fuzziness and probability[5].

However, what does fuzziness offer on vagueness? We can identify the distinction between fuzziness and vagueness with the following example taken from [1]:

> 'Bob will be back in a few minutes' is fuzzy, but 'Bob will be back sometime' is vague.

However, as pointed out in [2] "fuzziness formalizes reasoning under vagueness." In this example, we can define, a kind of 'metric' function, a.k.a truth function, that *measures* the distance between the time Bob came back and the time predicate "few minutes". We can define this function (or functions) by fixing only one point for "few minutes". It is clear that this function will have an infinite co-domain. So, for simplicity, the truth set will be fixed as [0, 1] in reals. Since the real interval [0, 1] and the set of real numbers have the same cardinality, we do not miss anything. However, this function

[4] We will spend an entire chapter on the methodological considerations about [LP].

[5] See [1].

cannot be defined for the sentence 'Bob will be back sometime', as we do not have a fixed (pre-determined) point to start with. *Sometime* is too vague to start with, hence we cannot formalize it in fuzzy logic.

Then, what does [LP] offer in terms of vagueness and fuzzy logic? As Hájek emphasizes in [2]:

> In fuzzy logic, the question is, what is the truth degree of a given formula in a given fuzzy structure.

The authors of [LP] follow this guideline throughout the whole paper. So, in order to make the liar sentence precise, the unary truth predicate Tr is taken as a fuzzy predicate. But, the arithmetic is kept as it was and in that way, we can formalize the liar formula. Therefore, the formal system we got is also consistent. As we underlined these accomplishments already, we will not repeat them.

The reviewer does not find it very interesting to make the liar sentence as having truth values in the real interval [0, 1]. But, the important achievement is the mathematical formalization of these concepts, and the further examination of the formal systems mentioned.

Similar conceptual approaches can be found easily in the literature. For example, the reviewer also wants to remark that Hájek considers fuzzy logic as a helpful tool for Sorites paradox [6].

Methodological Comments on [LP]

The reviewer had already spent many months to learn, discuss and write about Lakatos, particularly on [4]. Hence, the reviewer cannot help but to point one of the methodological problems of [LP] following Lakatos' approach on methodology of mathematics in the appendices of [4].

Obviously [LP] radically follows the Euclidean tradition (either intentionally or not) in the presentation of mathematical ideas. However, the Euclidean string of

$$\textit{definition} \longrightarrow \textit{lemma} \longrightarrow \textit{theorem} \longrightarrow \textit{proof} \tag{2}$$

does not let us grasp the underlying motivations and the reasons for the problems considered. Not only the reader was left alone in this mathematical marathon, but also s/he was not very well informed about the flourish of mathematical problems and their backgrounds.

The first obvious step that should be taken to prevent it was to break free from the string (2). Hence the appropriate way to start [LP] would be spending some time for the background of the problems considered. Not only it helps us understand the paper, but also it enables us to follow the footprints of the authors in the snowy paths of fuzzy thinking.

Conclusion

Although it is not methodologically clear from [LP] how the authors came up with the idea of extending PA by a infinitely many-valued truth pred-

[6] See [2]

icate Tr, the result is remarking. As it was pointed out already, a similar approach was used to analyze Sorites paradox in [2].

Therefore, the reviewer considers [LP] as one of the pioneering efforts to formalize the liar paradox, and hopes for further works on fuzzy approach to these paradoxes.

References

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[4] *Proofs and Refutations*, Imre Lakatos, Cambridge University Press, 1977.

[5] *Metamathematics of Fuzzy Logic*, review by F. J. Pelletier, The Bulletin of Symbolic Logic, vol. 6, No. 3, September 2000.