Top -of the- Logic

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There can never be surprises in logic.
Ludwig Wittgenstein

1 Prologue

1.1 Introduction and Motivation

This survey focuses on topology and its conceptual framework from a logical point of view. We are interested in how we can use logical tools to define some topological properties. But as a matter of taste, we will mainly follow Gabelaia's foot prints. However, it is not sufficient for our work to restrict our attention only to Gabelaia's works. So, we will spend some time to understand McKinsey and Tarski's basic proofs. Moreover, we will also focus on subset spaces (or let us better call them logic of knowledge) and try to explain the very basics of Dabrowski et al. and Georgatos' papers.

Hopefully, this investigation will help to make a picture of the current research on these topics while keeping the fundamental results in the background.

Nineteen-twenties and thirties witnessed Tarski and McKinsey's work on topological investigation of logic and algebras as emphasized in [10]. What motivated Tarski was the fact that propositional logic, Boolean algebras and topological spaces were each forming lattices. Thus, this observation facilitated the emergence of a new approach in logic. However, this emergence has some methodological importance. As it was claimed in [10] that,

> growth in mathematical knowledge often results when two or more related fields are unified by a hyphothesis of a partial structural analogy, which allows for the combination of their resources in the solution and discovery of problems.

Hence, we will trace back the mathematical growth in the field of knowledge spaces and modal logical interpretation of topology; and try to make clear the constructions that lead to the growth of knowledge in aforementioned fields. We will first start with basics and recall the definitions. Then, due to historical and mathematical importance, we will focus on McKinsey and Tarski's works.

Consequently, we will analyze applications of logical framework to topology in terms of definability results. Then, we will conclude with Moss and Parikh's and Georgatos' results. But first, let me sketch the history briefly.

1.2 Brief History

First let me recall the closure operator which has a significant role in our discussion. We call Clo closure operator if for all subsets A, B of our space, we have $Clo(\emptyset) = \emptyset$, $A \subseteq Clo(A)$, $Clo(A \cup B) = Clo(A) \cup Clo(B)$, $Clo(Clo(A)) = Clo(A)$

One of the very first applications of topology to (modal) logic is McKinsey's 1941 paper [11]. McKinsey, in this paper, by using the unary operator *closure*, forms a topological space and constructs the correspondence theorem between the logic of topological spaces and S4. As McKinsey himself emphasized in [11], McKinsey obtained a " ... *constructive method for deciding wheter an arbitrary given equation (involving symbols for the Boolean operators and for the topological operation of closure) is true in every topological space*" Recall that S4 is the logic that, among propositional axioms and K axiom, have the axioms $\Box\varphi \to \varphi$ and $\Box\varphi \to \Box\Box\varphi$. Later on, in 1944, McKinsey, together with Tarski, published a long and very detailed article [12] in which they clearly formalized the algebra of topological spaces. We will spend an entire chapter to discuss McKinsey and Tarski's results, so we leave the details to that section.

These two papers gave inspiration for the current researchers on this subject and in 1996, Moss and Parikh made use of topological framework to analyze modal effort and observation in [5]. Later on, Georgatos in [9] gave more constructions on this framework; however Georgatos sticks to the traditional treatment of knowledge, thus puts forward the idea that *a restriction of our view increases our knowledge.* We will follow their footsteps and investigate the logical structure with basic topological concepts.

As the more current research we should mention that, Dragalin worked on modal logics of geometrical structures in Euclidean spaces, Esakia worked on modal diamonds interpreted as topological derivative operator which also appeared in [12]. Kremer and Mints, on the other hand, studied Dynamic Topological Logic. Also interestingly, Mints and Zhang proved the topological completeness of S4 with respect to the real open interval $(0, 1)$.

2 Basics

In this section we will recall the fundamental definitions and results in topology. After recalling all the necessary information we will also include the results of McKinsey and Tarski and conclude with their fundamental results.

In order to observe similarities between modal logic and topology let us first start with the definitions and properties of topology.

Definition 2.1 (Topological Space). *$\mathcal{T} = (X, \Omega)$ is a topological space with a non-empty set X and a collection of its subsets Ω with the following properties:*

T1: Empty set and the set X itself are in Ω.

T2: Ω is closed under finite intersection.

T3: Ω is closed under arbitrary, possibly infinite, unions.

As usual we will call the elements of Ω *opens*. The complement of an open will be be called *closed*. Also, in order to have a relational semantics we will need some operations. $Int(A)$ will denote the biggest open subset of A, whereas $Clo(A)$ will denote the closure of A. More technical definitions and properties can be found in [6].

A *neighborhood* of x in X will be defined as any open set that contains x. Since there are too many neighborhoods of any x, we will somehow restrict them according to our interests. So, let us define $\Omega_x = \{O : O \in \Omega \text{ and } x \in O\}$.

Many different properties of topologies can be defined by additional rules and axioms. We will define Alexandroff Space and the connection between Definition 2.2 and Definition 2.4 will be made in the following pages.

Definition 2.2 (Alexandroff Space). *A topological space is Alexandroff if either of the following conditions are satisfied:*

- *Arbitrary intersections of opens are open.*
- *Every point has a least open neighborhood.*

Alexandroff spaces will play a central role in our discussions. First, we may obtain an open from the intersection of infinitely many opens. This obviously enlarges the property $T2$ of the Definition 2.1. Second property however will enable us to set a minimum neighborhood.

The next construction pretty much resembles the disjoint unions in modal logic.

Definition 2.3 (Disjoint Topological Sum). *For a family of disjoint topological spaces $\mathcal{T}_i = (X_i, \Omega_i)$ for $i \in I$, the (disjoint) topological sum is the topological space $\mathcal{T} = (X, \Omega)$ where $X = \bigcup_{i \in I} X_i$, and $\Omega = \{O \subseteq X : O \cap X_i \in \Omega_i, \text{ for all } i\}$.*

Disjoint topological sums will help derive some contradictory results in the proofs of definability theorems.

Now, we need to have a *closer* look to the closure operator as it will reveal some secrets of the topologies, and will be the primitive operation in our construction.

Definition 2.4 (Closure Operator). *Clo is a unary operator satisfying the following conditions for every O in the topological space:*

C1 $O \subseteq Clo(O) = Clo(Clo(O))$.

C2 $Clo(A \cup B) = Clo(A) \cup Clo(B)$.

C3 $Clo(\emptyset) = \emptyset$.

Further properties of closure operator can be found in [12] (or in any topology text book such as [6]. But, to make the concept of closure and closure algebras more clear let us consider the following example form [3]. Let $C^\infty(\mathbb{R})$ denote the set of the countable unions of convex subsets of $\mathbb{R}$. Recall that a subset S of $\mathbb{R}$ is convex if for all $x, y \in \mathbb{R}$, $x \leq z \leq y$ implies $z \in S$. Now, consider the Boolean algebra generated by $C^\infty(\mathbb{R})$ and Clo operator, and denote it $B(C^\infty(\mathbb{R}), Clo)$. Then $B(C^\infty(\mathbb{R}), Clo)$ is a closure algebra. It is very easy to see that Clo satisfies the properties (C1), (C2) and (C3) on $B(C^\infty(\mathbb{R}), Clo)$. [1]

Similar definitions can be constructed for Int operator. As a general concept, we will define the following:

Definition 2.5 (Interior Algebra). *An Interior Algebra is a pair $(B, \Box)$ where B is a Boolean algebra and $\Box$ is an operator which assigns each element a of B to an element $\Box a$ such that the following are satisfied:*

$\Box(a \wedge b) = \Box a \wedge \Box b$.

$\Box\Box a = \Box a \leq a$.

$\Box\top = \top$.

It is not difficult to see that interior operator is the dual of closure operator. We will make it clearer when we need them. The trivial example for an interior algebra is the full set algebra over any topological space with Int operator.

Now let us define the most important and famous properties of topological spaces. We will present the definability results based on these definitions.

Definition 2.6 (Connectedness). *A topological space $\mathcal{T} = (X, \Omega)$ is connected if X cannot be represented as the disjoint union of two open sets.*

A topological space $\mathcal{T} = (X, \Omega)$ is well-connected if $X = U \cup V$ implies $X = U$ or $X = V$ for U, V in Ω.

Definition 2.7 (Extremally Disconnectedness). *A topological space $\mathcal{T} = (X, \Omega)$ is extremally disconnected if either of the following conditions are satisfied:*

- *The closure of any open is open.*
- *The closures of any two disjoint opens are disjoint.*

[1] In [3], Bezhanishvili and Gehrke proves the completeness of S4 with respect to $B(C^\infty(\mathbb{R}), Clo)$. But in this work we will not present the completeness proofs.

Definition 2.8 (Compactness). *A topological space $\mathcal{T} = (X, \Omega)$ is compact if any family $(C_i)_{i \in I}$ of closed subsets of X with the finite intersection property has a non-empty intersection.*

Definition 2.9 (Frontier). *For a topological space $\mathcal{T} = (X, \Omega)$ and for a subset A, define the frontier set for A to be the set $Clo(A) \cap Clo(-A)$ and denote with $Fr(A)$. The topological space is called atomic if the frontier of any subset has an empty interior, that is $Int(Fr(A)) = \emptyset$.*

Now we can define the topological model and truth. First let us fix the language first. Our basic modal language $\mathcal{L}$ for topological spaces will consist of countable number of proposition letters p_i, the truth constant $\top$, unary connective $\neg$ and binary connective $\wedge$ and the modal operator $\Box$. Modal formulas ϕ will be built in the following way:

$$\phi := p_i \mid \top \mid \alpha \wedge \beta \mid \Box \alpha$$

where α and β are formulas.

We will let $\Diamond$ be the dual of $\Box$ such that $\Diamond \phi = \neg \Box \neg \phi$. We will include the $\Diamond$ in our truth definition to make it clear and precise.

Definition 2.10 (Topological Model and Truth). *$\mathcal{M} = (X, \Omega, V)$ is called a topological model where (X, Ω) is a topological space and V is a valuation. The truth definitions for the point x in X go as follows:*

$\mathcal{M}, x \models p$	*iff*	$x \in V(p)$.
$\mathcal{M}, x \models \top$		*always.*
$\mathcal{M}, x \models \phi \wedge \psi$	*iff*	$\mathcal{M}, x \models \phi$ *and* $\mathcal{M}, x \models \psi$.
$\mathcal{M}, x \models \neg \phi$	*iff*	$\mathcal{M}, x \not\models \phi$.
$\mathcal{M}, x \models \Box \phi$	*iff*	$\exists O \in \Omega$ *such that* $x \in O$ *and* $\forall y \in O$ *we have* $\mathcal{M}, y \models \phi$.
$\mathcal{M}, x \models \Diamond \phi$	*iff*	$\forall O \in \Omega$ *such that* $x \in O \rightarrow \exists y \in O$ *such that* $\mathcal{M}, y \models \phi$.

Now, let us extend the valuation function to grasp the topological notions. We will make it compatible with the topological space set up.

Definition 2.11 (Properties of Valuation). *For the valuation V of the topological model $\mathcal{M} = (X, \Omega, V)$, we define:*

$V(p) \subseteq X$ *for all propositional variables* p.

$V(\neg \phi) = X - V(\phi)$.

$V(\phi \vee \psi) = V(\phi) \cup V(\psi)$.

$V(\phi \wedge \psi) = V(\phi) \cap V(\psi)$.

$V(\phi \rightarrow \psi) = (X - V(\phi)) \cup V(\psi)$.

$V(\Box \phi) = Int(V(\phi))$.

$V(\Diamond \phi) = Clo(V(\phi))$.

Now we will review modal logic axioms and define the corresponding logics:

Definition 2.12 (Modal Axioms and Modal Logics S4, S4.1 and S4.2)**.** *Let the modal axioms be defined as:*

N $\Box\top$

T $\Box\phi \to \phi$

R $\Box(\phi \wedge \psi) \leftrightarrow (\Box\phi \wedge \Box\psi)$

4 $\Box\phi \to \Box\Box\phi$

.1 $\Box\Diamond\phi \to \Diamond\Box\phi$

.2 $\Diamond\Box\phi \to \Box\Diamond\phi$

The ***Modal Logic S4*** *is the smallest set of modal formulas which contains all the propositional tautologies and N, T, R and 4 axioms, and is closed under modus ponens, substitution and monotonicity (i.e. from $\phi \to \psi$, derive $\Box\phi \to \Box\psi$).*
The Modal Logic S4.1 *is the extension of S4 with the additional axiom .1.*
The Modal Logic S4.2 *is the extension of S4 with the additional axiom .2.*

These axioms might easily be formulated in a different way. For example axiom (R) might have been stated as $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$. However, the way we do it now will help us associate these axioms to topological framework.

Now let us consider the mathematical results that we can obtain from aforementioned definitions and properties. We will start with the results of McKinsey and Tarski underlying the mathematical importance of their results.

3 Results of McKinsey and Tarski

McKinsey and Tarski's works are historically very important without doubt. But, despite the historical importance, we will focus on their results of McKinsey and Tarski due to its mathematical strength. However, although these results are fundamental, we will not depend on them.

In a very early paper [12], McKinsey and Tarski introduced $\Diamond$ operator as Clo operator. They proved an astonishing result. We liked that theorem so much that we can directly copy here:

Theorem 3.1. *Every closure algebra is isomorphic with a subalgebra of the closure algebra over a topological space (in the strict sense).*

Before presenting the proof, let us observe the earlier results that McKinsey and Tarski obtained. Hopefully, these observations will help us grasp the fundamental notions that McKinsey and Tarski utilezed in Theorem 3.1. Moreover, taking this theorem as a mathematical invention, we will able to present the logic of this discovery in a very naive sense.

We start with an easy observation. It is now very well known that, topological spaces form a closure algebra with respect to Clo operator. The proof easily follows from the definitions of topological spaces and closure algebras. Hence, we will skip it.

The first important connection between topological spaces and closure algebras are established with a theorem which states that *every closure algebra is isomorphic with a subalgebra of the closure algabra over a topological space in the wider sense.* But, what is the difference between *strict sense* and *wider sense*. The answer comes in [12], if we assume $Clo(\emptyset) = \emptyset$ we obtain a topological space in wider sense, but instead if we assume the property

$$A \subseteq S, \text{ and } A \text{ contains at most one point, then } Clo(A) = A \tag{1}$$

we obtain topological space in strict sense. We wish to underline that, in the topological spaces in strict sense, any finite set is equal to its own closure. This result follows from the Equation (1) and the fact that closure of the union of two sets is the union of the closures of both sets. But it is not necessarily the case in the topological spaces as the condition $Clo(\emptyset) = \emptyset$ is weaker than Equation (1).

Now, let us go back to the proof of this statement. The proof uses the Stone's Representation Theorem to get the fact that the given closure algebra is isomorphic to a Boolean algebra with a field of sets. Recall that Stone's theorem basicly states that any boolean algebra is isomorphic to a set algebra. Then, it is not difficult to establish the isomorphism between the Boolean algebra and closure algebra. For every member in the given closure algebra find the corresponding member in the set algebra. (Remember that, this is due to Stone's Theorem) So, for x, y in the closure algebra $\mathcal{K}$, we have corresponding sets X, Y in the Boolean algebra $\mathcal{R}$. So, $Clo_{\mathcal{K}}(x) = y$ and $Clo_{\mathcal{R}}(X) = Y$ hold, where $Clo_{\mathcal{K}}$ denotes the closure operator in the closure algebra $\mathcal{K}$ and $Clo_{\mathcal{R}}$ denotes that of the Boolean algebra $\mathcal{R}$. Recall that our Boolean algebra was defined on a field of sets, say $\{S_i\}$ is the collection of these sets. So take this collection, construct their union $S = \bigcup\{S_i\}$ and consider the power set of S. Now we obtained a power set Boolean algebra. Then we can extend the $Clo_{\mathcal{R}}$ operator to the power set space such that $Clo_{\mathcal{R}}(X) = Clo_{\wp(\mathcal{R}')}(X)$ is satisfied for sets X where $\mathcal{R}'$ is the power set Boolean algebra. So, we obtained the topological space with respect to the union set S and to the closure operator $Clo_{\wp(\mathcal{R}')}$ defined on the power set of S. This finishes the proof.

Secondly McKinsey and Tarski proves that, *for any topological space $\mathcal{S}$ in wider sense, there is a (strict sense) topological space $\mathcal{S}'$, such that the closure algebra over $\mathcal{S}$ is isomorphic to a subalgebra of the closure algebra of $\mathcal{S}'$*. To prove that, McKinsey and Tarski define a isomorphism h such that $h(x) \cap h(y) = \emptyset$ for $x, y \in \mathcal{S}$ and the proceed to prove that h is the desired function and it satisfied the requirements set in the statement of the proof. Recall that, the condition $h(x) \cap h(y) = \emptyset$ establish the well-connectedness property of the space.

All the machinery that were developed hitherto will serve as a necessary proto-theorems for the proof of Theorem 3.1. Hence, now we can conclude the proof by combining previous two results.

We also want to remark that, we can state the Theorem 3.1 with more contemporary terminology as follows:

Theorem 3.2. *Every finite well-connected topological space is an open image of a metric seperable dense-in-itself space.*

Moreover, this result was strengthened by showing that every finite space is an open image of a connected metric seperable dense-in-itself space. Thus, it was proved that S4 is complete with respect to both the real line and the rationals as well. Alternative proofs of the completeness of S4 with respect to topological spaces can be found in [1] and [3].

Moreover, topological **derivative algebras** were defined by McKinsey and Tarski in [12] as well. Since the derivative operator cannot be defined in terms of closure operator, new operator and new algebra, namely derivative algebras, were introduced. More than thirty years later, L. Esakia interpreted the diamond operator as the topological derivative operator and menaged to give an axiomatisation of the base logic, however these results were published only in 2001. An extensive treatment of the modal logic of the topological derivative operation can be found in a very recent paper [2]. Yet, we will present a brief outline of the derivative algebras.

We will start by recalling the fact that, for every topological space we have $Clo(A) = A \cup d(A)$ for any $A \subseteq X$ where d stands for the derivative operator. By duality it is easy to see $Int(A) = A \cap t(A)$ where $t(A)$ is the set of the limit points. Now let us denote the t operator by $\Box_d$, and d operator by $\Diamond_d$.

Now we can establish a *translation* between Clo logic and d logic. To do that just replace $\Box_{Clo}\phi$ with $\phi \wedge \Box_d\phi$. Observe that translation does not effect Boolean cases (by the definitionof the translation). So let us consider the non-trivial case. Let $X \models_{Clo} \Box_{Clo}\phi$. That is $Int(V(\phi)) = X$. Then by the translation $V(\phi) \cap t(V(\phi)) = X$ which means $\models_d \phi \wedge \Box_d\phi$. This completes the proof. For deeper and more detailed results reader might consult to [2].

4 Applications of Logical Framework to Topology

In this section we will focus on topology and utilize topological reasoning to logic. We will especially concentrate on definability results.

However, the main motivation for using topological reasoning in logic is the fact that closure operator in topology forms a lattice, as propositional logic does. So, let us start with the basic definitions before proceeding to the definability results.

The very first question we want to ask to the reader is,

Question: What is the relation and (if exists) correspondence with opens in topology and formulas (and sentences) in logic?

Keeping this question in mind will help the reader to keep track of the following constructions easily.

4.1 Definability Results

After recalling all the necessary (but not sufficient) information about topological spaces, now we will apply modal logical tools to topology.

The very first observation we will make is on the correspondence between S4 and topological spaces.

Theorem 4.1. *The theorems of S4 are valid on every topological spaces.*

Proof. First, for the sake of clarity let us recall the properties of Int operator. Similar to the properties of Clo operator in Definition 2.4, we have

I1 $Int(O) \subseteq O$.

I2 $Int(A \cap B) = Int(A) \cap Int(B)$.

I3 $Int(X) = X$

I4 $O = Int(Int(O))$

It is now easy to see (I1), (I2), (I3) and (I4) correspond to (T), (R), (N) and (4) axioms respectively.

Modus ponens is preserved by the property T3 of Definition 2.1. Monotonicity is also preserved since, if we have $\phi \to \psi$ corresponding to $A \subseteq B$ in the topological space then, it is very easy to see $Int(A) \subseteq Int(B)$, that is $\Box\phi \to \Box\psi$, by using the property I2 above. Clearly, substitution rule is preserved as well. $\Box$

Now we can take a look at what we can define in topology using modal logic.

Theorem 4.2. *S4.2 defines the class of extremally disconnected topological spaces.*

Proof. Considering Theorem 4.1, we only need to show that (.2) axiom defines the extremally disconnectedness. We will simplify the proof in [7].

Take an open O and consider its closure $Clo(O)$. Since O is open $Int(O) = O$ and $Clo(O) = Clo(Int(O))$. This expression corresponds to $\Diamond\Box\phi$ for some ϕ. But we have (.2), then by modus ponens we have $\Box\Diamond\phi$, which is $Int(Clo(O))$. But, by definition $Int(S)$ is open for every set S, we are done. Hence, closure of every open is open, and S4.2 defines the class of extremally disconnected topological spaces.

For the converse direction we will use the contrapositive. Assume (.2) is not valid. Then, for any open set O in an extremally disconnected topological spaces, we have that closure of O is open. Since the closure of O is open, it is equal to its interior. So, we obtained $\Diamond\Box\phi$. Since we have (.2) invalid, we cannot have $\Box\Diamond\phi$, that is interior of a closure of a set. So, the closure of the set O, is not equal to its interior. Thus, our topological space is not extremally disconnected.

This completes the proof. $\Box$

The completeness of S4.2 with respect to class of extremally disconnected topological spaces can also be proven, but we will skip completeness proofs in this work in order to focus more on definability.

We have already defined atomic spaces, and now we will establish the connection between atomic spaces and S4.1.

Theorem 4.3. *S4.1 defines the class of atomic spaces.*

Proof. If the topological space $\mathcal{T}$ realizes McKinsey axiom (.1) then, we have $Int(Clo(O)) \subseteq Clo(Int(O))$, for any subset O.

So, $Int(Clo(O)) \cap Clo(Int(O)) \equiv Int(Clo(O)) \cap - - Int(- - Clo(-O)) \equiv Int(Clo(O)) \cap Int(Clo(-O)) \equiv Int(Clo(O) \cap Clo(-O)) \equiv Int(Fr(O)) = \emptyset$, which means $\mathcal{T}$ is atomic. $\Box$

The completeness of S4.1 with respect to class of atomic spaces can also be proven, but we will again skip completeness proofs in this work in order to focus more on definability.

As a last remark, we wish to underline that, modal formulas are preserved under disjoint topological sums. This resembles the corresponding theorem in modal logic. Proof is straight-forward and left as an exercise.

Now let us consider the relations between Alexandroff spaces and qo-sets (sets with a reflexive and transitive relation). The second condition of the definition of Alexandroff spaces, i.e. Definition 2.2, let us find a correspondence between qo-sets and the Alexandroff topological spaces built on this sets.

How to define a reflexive and transitive relation out of an Alexandroff space? Simply, we can define xRy holds if and only if y is in the least open neighborhood of x. Recall that, the least open neighborhood exists because of the definition of Alexandroff spaces. It is very easy to see the reflexivity and transitivity of R.

Conversely, let us have a qo-set X with the relation R. Consider the upward closed sets of X. Recall that $A \subseteq X$ is upward closed if $w \in A and wRv$ implies $v \in A$. We can form a topology over the upward closed subsets of A, since R is reflexive, it will have a least neighborhood, and arbitrary intersection of opens will be open and will be in the topology because of the transitivity of the relation R.

Now we will make the connection between qo-set X, and the Alexandroff space obtained from the upward closed subsets of X.

Theorem 4.4 (Frame - Space Correspondence). *Let $\mathcal{F} = (X, R)$ be a modal frame where R is reflexive and transitive, and $\mathcal{T} = (X, \Omega)$ is a corresponding Alexandroff space. Then:*

$$\mathcal{F}, V, x \models \phi \text{ if and only if } \mathcal{T}, V, x \models \phi$$

for any valuation V and any point x.

Proof. The proof goes by induction and hence is skipped. But, for the $\Box\varphi$ case recall that, "$\Box\varphi$ is true at w" means that φ is true at all nearby points of w, and successors of any element under R constitutes the least open neighborhood of this point. But remember also the second property of Alexandroff Spaces from Definition 2.2. Thus, the proof is straight forward. □

We might recall that in topology, some certain functions preserves some certain properties between topological spaces. So, now let us investigate if we might have some similar results. Recall that, a function f is *continuous*, if the inverse image of any open is open. Moreover, f is *open* if the it sends opens to opens. We will call both open and continuous functions *interior*. Interior maps play a central role in modal satisfaction in topological spaces.

Theorem 4.5. *It $\mathcal{T}_1$ and $\mathcal{T}_2$ are topological spaces and f is an interior map from $\mathcal{T}_1$ onto $\mathcal{T}_2$, then for any modal formula we have $\mathcal{T}_1 \models \phi$ if and only if $\mathcal{T}_2 \models \phi$.*

Proof. To prove the theorem let us fix the valuation as $V_1(p) = f^{-1}(V_2(p))$. Proof is by induction on the complexity of ϕ. Boolean cases are easy, so let us consider the modal case. Let us have $\mathcal{T}_1, V_1, x \models \Box\phi$, then $\mathcal{T}_1, V_1, O_x \models \phi$ follows by definition,where O_x is an open neighborhood of x. Since O_x is open and f is an open map, we have $f(O_x)$ open. By induction hypothesis we get, $\mathcal{T}_2, V_2, f(O_x) \models \phi$. Then, again by definition of the modal operator $\Box$, we get, $\mathcal{T}_2, V_2, f(x) \models \Box\phi$.

For the converse direction, we first want to point out that f is given surjective. Similarly, assume $\mathcal{T}_2, V_2, f(x) \models \Box\phi$. By definition, then we have $\mathcal{T}_2, V_2, O_{f(x)} \models \phi$. As $O_{f(x)}$ is open and f is continuous we have $f^{-1}(O_{f(x)})$ open in $\mathcal{T}_1$. By the induction hypothesis, we get $\mathcal{T}_1, V_1, O_x \models \phi$. Definition of the modal operator gives the result: $\mathcal{T}_1, V_1, x \models \Box\phi$ Note, that the validity is preserved by the definition. □

This concludes our discussion on definability results.

4.2 Non-definability Results

Now let us observe what we cannot define. We will use the standard tricks from modal logic to prove what cannot be defined. We will use the fact that disjoint topological sums are validity preserving operations.

To see this fact, take a family of disjoint topological spaces $\{\mathcal{T}_i\}_{i \in I}$. We claim, if *each* disjoint topological spaces $\mathcal{T}_i$ satisfies a formula then, the disjoint topological sum $\mathcal{T} = \bigcup_{i \in I} \mathcal{T}_i$ satisfies the same formula as well. to get a contradiction assume $\mathcal{T} \nvDash \phi$ for some formula ϕ. Then for some valuation V and for a state x, we have $\mathcal{T}, V, x \nvDash \phi$. We can easily define the valuation for X_i as follows, $V_i(p) = V(p) \cap X_i$. As $x \in X_i$ for some i, we will have for some i, $\mathcal{T}_i, V_i, x_i \nvDash \phi$. We leave it as an exercise to show that $\mathcal{T}, V, x_i \models \phi$ if and only if $\mathcal{T}_i, V_i, x_i \models \phi$ for all formula ϕ. So, we have $\mathcal{T} \nvDash \phi$. This concludes the argument.

However, the converse direction is also easily established. If the topological sum $\mathcal{T}$ satisfied a formula, then this formula is satisfied in each member $\mathcal{T}$ of the family of topological spaces. Thus, we can conclude that disjoint topological sum is a validity preserving operation.

We would like to pay attention to the similarity of the arguments between disjoint topological sums and disjoint unions in modal logic. In the latter case, we consider family of models in the same modal similarity type, and construct the disjoint union in a very similar vein. We also have a very similar conclusion in modal case, that is, modal satisfaction is invariant under disjoint unions. This explains how and why we use the methods from modal logic to prove non-definability results. Non-satisfaction proofs will follow the same methodology.

Now, by using disjoint topological sums we will prove that the class of connected topological spaces and the class of compact topological spaces cannot be defined in modal language.

Theorem 4.6. *The class of connected topological spaces and the class of compact topological spaces are not modally definable.*

Proof. We will present a counter model for both cases following [7]. Consider the disjoint topological sum of singletons $X_i = \{i\}$ where $i \in I$. For each X_i, we have the corresponding topology $\Omega = \{\emptyset, X_i\}$. Then, clearly each topological space $\mathcal{T}_i = (\Omega_i, X_i)$ is compact and connected. Now assume I countably infinite. Then, disjoint topological sum $\mathcal{T} = \bigcup_{i \in I} \mathcal{T}_i$ is not connected and is not compact.

It is not compact because under the finite intersection property, the closed sets in the disjoint topological sum has an empty intersection. So $\mathcal{T}$ is not compact.

It is not connected neither: take any X_i with its compliment. They are disjoint opens but their union forms universe.

But, under disjoint topological sums, the properties should have been preserved. So, we cannot express compactness and connectedness in modal language. $\square$

However, to be able to define connectedness, we have to extend the modal language with the universal modality U, where $U\phi$ means ϕ holds in all worlds of the model; and with the dual operator E, where $E\phi$ means ϕ holds in at least one world. Then we can express connectedness as $U(\Diamond\phi \to \Box\phi) \to (U\phi \vee U\neg\phi)$. For further remarks on the extended modal language, reader should consult to [1].

At this point, we have some further results:

Corollary 4.1. *Disconnectedness is not modally definable.*

Proof. Similiar to what we did for connected spaces. One point space is connected, although the disjoint topological sum is not. $\Box$

Corollary 4.2. *Non-compactness is not modally definable.*

Proof. To get a contradiction assume there exists a formula that defines non-compactness in modal language. Consider the reals with their standard topology and $X = \{0, 1\}$ with the interior map f sending rationals to 0, irrationals to 1. Any finite set is compact so is X. Yet reals is not compact. So f cannot preserve the property which was supposed to define non-compactness in modal language. $\Box$

The whole constructions hitherto presented can be concluded with a nice theorem on modal definability. It should be recalled that in Kripke semantics, Goldblatt-Thomason Theorem defines which kind of first-order frame classes are modally definable and, more importantly which are not. Thus, Goldblatt-Thomason Theorem gives the necessary and sufficient conditions for modal definability, that is a first order definable class is modally definable if it is closed under taking bounded morphic images, generated subframes, disjoint unions and reflects ultrafilter extensions.

Definition 4.1 (Alexandroff Extension)**.** *For a given topological space $\mathcal{T} = (X, \Omega)$, its Alexandroff extension is the Alexandroff space $\mathcal{T}* = (\wp(X), \Omega)$ of the interior algebra of all subsets of X.*

Theorem 4.7 (Topological Goldblatt-Thomason Theorem)**.** *The class K of topological spaces which is closed under formation of Alexandroff extensions is modally definable if and only if it is closed under taking subspaces, interior images, (disjoint) topological sums and it reflects Alexandroff extensions.*

Proof. The proof is omitted as it requires much more mathematical construction. Interested reader is kindly referred to [7]. $\Box$

The class of Alexandroff spaces is not modally definable as they do not reflect Alexandroff extensions in the way they were defined.

Corollary 4.3. *The class of Alexandroff spaces is not modally definable.*

This concludes our discussion on non-definability results.

5 Knowledge Spaces and Topological Approach

In this part, we will focus on the applications of topological reasoning in logic, specifically in logics of knowledge.

We will start with definitons as usual and try to expose the results that Georgatos and Dabrowski et al. obtained. We picked their works as they present a great importance for the very current research in this topics. So, we will briefly give the outline of their well know papers and accomplishments. However, in order to keep focused, we will skip completeness results.

5.1 Definitions and Background

We will use two different modalities for our system. K will stand for knowledge whereas $\Box$ will stand for the notion of effort[2]. So let us begin with the usual truth definitions. To keep the following definitions distinct from the ones in the previous chapter, we will adopt a different notation.

Definition 5.1 (Satisfiability in Knowledge Spaces). *A subset frame is a pair $\mathcal{X} = (X, \mathcal{O})$ where X is a set of points and $\mathcal{O}$ is a set of non-empty subsets of X that are called opens. A subset space space $\mathscr{X} = (\mathcal{X}, v)$ is a triple where $\mathcal{X}$ is a subset frame and $v : At \longrightarrow \wp(X)$ is a valuation function from the set of atomic sentences to the power set of the set X. For all p in X, and $p \in U \in \mathcal{O}$, we define the satisfaction relation inductively as follows:*

$p, U \models A$	*iff*	$p \in v(A)$
$p, U \models \phi \wedge \psi$	*iff*	$p, U \models \phi$ *and* $p, U \models \psi$
$p, U \models \neg\phi$	*iff*	$p, U \models \phi$
$p, U \models K\phi$	*iff*	$q, U \models \phi$ *for all* $q \in U$
$p, U \models \Box\phi$	*iff*	$p, V \models \phi$ *for all* $V \in \mathcal{O}$ *such that* $p \in V \subseteq U$

So the, $\Box$ operator corresponds to shrinking an open while keeping the reference point whereas K refers to moving inside the given open. Thus, we define the duals of $\Box$ and K as $\Diamond$ and L respectively.

$p, U \models L\phi$	iff	$\exists q \in U$ such that $q, U \models \phi$
$p, U \models \Diamond\phi$	iff	$\exists V$ such that $p \in V \subseteq U$ and $p, V \models \phi$

Now, we can present the axiomatization which was shown to be complete in [5]. The axioms are

i. Instances of the tautologies of propositional logic

ii. $(A \to \Box A) \wedge (\neg A \to \Box\neg A)$ for atomic A

iii. $K\phi \to (\phi \wedge KK\phi)$

iv. $\Box\phi \to (\phi \wedge \Box\Box\phi)$

v. $K(\phi \to \psi) \to (K\phi \to K\psi)$

[2]We will discuss *effort* soon.

vi. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

vii. $L\phi \to KL\phi$

viii. $K\Box\phi \to \Box K\phi$

The rules of inference will be Modus Ponens, K-Necessitation (from ϕ derive $K\phi$) and $\Box$-Necessitation (from ϕ derive $\Box\phi$).

We will make some observations on these axioms in the following section.

5.2 Definability Results

Now, we can start to review the definability results. First, note that $v(A)$ is open if and only if $A \to \Diamond KA$ is valid in the model, where A is an atomic predicate. To see that, recall that a set is open if and only if every point in the set has an open neighborhood which lies entirely in the set.

So, $v(A)$ is open if and only if for all points in A there is a neighborhood for each of those points (corresponds to $\Diamond$) such that, those neighborhoods entirely lie in the set (corresponds to K). Using duality we get similar results for closed sets. So, $v(A)$ is closed if and only if $\Box LA \to A$ is true in the model.

In the same vein let us look for the expressions for dense sets. We know that $v(A)$ is dense if and only if each nonempty subset of the universe has an element of $v(A)$ (See [6]). Hence, $v(A)$ is dense if and only if LA is valid. Consequently, it is nowhere dense if and only if $\Diamond L\neg A$ is valid in the model. These results depend on the fact that we are in a topological spaces considering the axioms given.

Now, let us have a closer look at S4. Aiello et al. in [1] presents a very interesting result: *S4 has the effective finite model property with respect to the class of topological spaces.* Although S4 is the logic of the class of all topological spaces which are compact and dense-in-itself (due to McKinsey and Tarski), the resulting finite model may not be dense-in-itself. They also proved that, S4 is complete with respect to the open real interval $(0, 1)$, and hence with respect to $\mathbb{R}$ both with the usual topology. Moreover, they prove that *a finite frame is rooted if and only if the corresponding topological space is well-connected.* A similar statement can be found in [3] as follows: *S4 is complete with respect to the closure algebras over finite quasi-trees*.

5.3 Results of Georgatos and Moss & Parikh

The very first contribution that Moss and Parikh presented was the introduction of the notion of *effort* to logic of knowledge. However we can trace back this notion to Vicker's book [13] where he defines the affirmative assertions with the notion of *affirmability*. An assertion is affirmative if and only if it is *true in the circumstances when it can be affirmed*. This definition implicitly makes use of the concept of effort. If after some effort we can affirm that a assertion is true then it is true. So for the affirmative statement A, we will have $A \to \Diamond KA$. Similarly for closed sets, we will have the *refutative* statements which satisfy $\Box LA \to A$.

Recall that an assertion is refutative is an assertion such that if it does not hold, then it is possible to know it (by some effort). It is also clear that, affirmative statements are closed under infinite disjunctions whereas refutative statements are closed under infinite conjunctions.

Georgatos constructs a system MP* named after Moss and Parikh. He added two additional axioms (i.e. 11. and 12. below) to get his system from MP of Moss and Parikh. So, now let us list his axioms as they are slightly different from the list of axioms in the previous section.

1. All propositional tautologies.
2. $(A \to \Box A) \land (\neg A \to \Box \neg A)$ for A in the countable set of atomic formulae.
3. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$.
4. $\Box\phi \to \phi$.
5. $\Box\phi \to \Box\Box\phi$.
6. $K(\phi \to \psi) \to (K\phi \to K\psi)$.
7. $K\phi \to \phi$.
8. $K\phi \to KK\phi$.
9. $\phi \to KL\phi$.
10. $K\Box\phi \to \Box K\phi$.
11. $\Diamond\Box\phi \to \Box\Diamond\phi$.
12. $\Diamond(K\phi \land \psi) \land L\Diamond(K\phi \land \zeta) \to \Diamond(K\Diamond\phi \land \Diamond\psi \land L\Diamond\zeta)$.

It was porven in [5] that Axioms 1 - 10, with modus ponens, K-necessitation and $\Box$-necessitation are sound and complete with respect to subset spaces. Axiom 11 is familiar actually, it is .2 axiom that we already defined. Dabrowski et al. call it *Weak Directedness Axiom*.

The last axiom defines the class of subset spaces that are closed under finite union and intersection. In order to make this definability result clearer we will observe wheter the assumptions (i.e. closure under finite union and intersection) are needed in the proof of definability. But, we will skip the first five axioms as it is very obvious that they do not need these assumptions. The interested reader might check the soundness of K logic in [4] to see the detailed procedure. So let us start with the axioms that contains the new modalities.

Axiom 6: Let $p, U \models K(\phi \to \psi)$. Then by Definition 5.1 we have $q, U \models \phi \to \psi$ for all $q \in U$. To show $p, U \models K\phi \to K\psi$, assume $p, U \models K\phi$ which means $q, U \models \phi$ for all $q \in U$. Then by modus ponens for all $q, U \models \psi$ for all $q \in U$, which means $p, U \models K\psi$.

Axiom 7: Let $p, U \models K\phi$ then by definition $q, U \models \phi$ for all $q \in U$. Take q as p and result follows.

Axiom 8: Let $p, U \models K\phi$ then by definition $q, U \models \phi$ for all $q \in U$. As $q, \models \phi$ holds for all $q \in U$, we will have $r, U \models \phi$ for all $r \in U$ as r is a dummy variable. Then going backwards we will have $q, U \models K\phi$ and $p, U \models KK\phi$.

Axiom 9: Let $p, U \models \phi$. Take an arbitrary point q in U (it does not matter if q satisfies ϕ or not). Then by definition of L operator above we have $q, U \models L\phi$ for all $q \in U$. Since q was arbitrary, we obtain $p, U \models KL\phi$ as for all q in U, we have $q, U \models L\phi$.

Axiom 10: Let $p, U \models K\Box\phi$. Then by definition we have $q, U \models \Box\phi$ for all $q \in U$ and $q, V \models \phi$ for all subsets $q \in V \subseteq U$. To show $p, U \models \Box K\phi$ let $p \in V' \subseteq U$. Then, we should show $p, V' \models K\phi$, that is $q', V' \models \phi$ for $q' \in V' \subseteq U$. As for all q in U, we have $q', U \models \Box\phi$, as we also have for all q in $V \subseteq U$, $q, V \models \phi$, the result $q', V' \models \phi$ follows.

Axiom 11: Assume $p, U \models \Diamond\Box\phi$. Then by definition, there exists a subset V such that $p \in V \subseteq U$ and $p, V \models \Box\phi$. Again by definition, we get $p, W \models \phi$ for all W such that $p \in W \subseteq V$. Now let us go backwards. We get $p, V \models \Diamond\phi$ as W was arbitrary and $p \in W \subseteq V$. By similar reasoning, we get $p, U \models \Box\Diamond\phi$ as $V \subseteq U$ and $p \in V$.

Axiom 12: Let $p, U \models \Diamond(K\phi \wedge \psi) \wedge L\Diamond(K\phi \wedge \zeta) \rightarrow \Diamond(K\Diamond\phi \wedge \Diamond\psi \wedge L\Diamond\zeta)$. As $p, U \models \Diamond(K\phi \wedge \psi)$ there exists $U_p \subseteq U$ such that $p, U_p \models K\phi \wedge \psi$; and similarly there exists $q \in U$ and $U_q \subseteq U$ such that $q, U_q \models K\phi \wedge \zeta$. As the space is closed under unions we have $p, U_p \cup U_q \models K\Diamond\phi$ and $q, U_p \cup U_q \models K\Diamond\phi$ and $p, U_p \cup U_q \models K\Diamond\psi$ and $p, U_p \cup U_q \models K\Diamond\zeta$. Then the reslut follows, $p, U_p \models \Diamond(K\Diamond\phi \wedge \Diamond\psi \wedge L\Diamond\zeta)$. Notice that, in this proof we used the assumption of closure under unions.

In this context, Georgatos also showed that MP* is characterized by closed topological frames.

After giving the definitions we gave already, Dabrowski et al. on the other hand, demonstrated an interesting property of subset spaces, that is, subset spaces do not have the finite model property contrary to Kripke semantics. For this they build a (counter) model by explicitly stating its universe, and find a formula which can never have a finite model. But, also we want to remark that, subset spaces are decidable, although they lack the finite model property. Then they proceed to completeness proof. As it is quite well known, for the completeness proof, maximal consistent sets should be used, and for this purpose, the truth lemma should be established first . So, they form the truth lemma which is based on the some auxialiry functions with the help of L and $\Diamond$ operator. Then they proceed to the proof of completeness.

Theorem 5.1 (Completeness of Subset Space Logic)**.** *The axioms that we presented in the preceeding section are complete for interpretation in subset spaces.*

We have to skip the technical proof as it requires a large amonut of auxiliary construction. The interested reader is referred to [5].

Georgatos, on the other hand, bases his treatment to the following fact: *A restriction of our view increases our knowledge*. It is because if we had a smaller set of possible knowledge, then we would be able to have accurate knowledge more easily. However, things are not as simple as they seem. If we want an increase in our knowledge in a logic which is closed under deduction; we surely need an external information that does not lie in our system. On the given topology Georgatos shows that, it is possible to find a partition that the given formula keeps its truth value in the points of this partition. Then with some sophisticated mathematical constructions he proves the finite satisfiability theorem.

Theorem 5.2 (Finite Satisfiability Theorem)**.** *If ϕ is satisfied in any topological space, then it is also satisfied in a finite topological space.*

The finite space is simply a quatient of the given one under two equivalence operations. The first equivalence relation was constructed on the opens wheras the second one was constructed on the points of the topological space. Moreover, the size of the finite topological space is determined by a function of the complexity of the given formula ϕ. So, in order to test the validity of a given formula ϕ we will surely have a finite number of topological spaces. So, the next result simply follows:

Theorem 5.3 (Decidability Theorem)**.** *The thory of finite topological spaces is decidable.*

This conclusion ends our discussion on the logic of subset spaces.

6 Epilogue

We basicly covered two different approaches. However, they have some important differences. The first approach, first of all, has one modal operator. This sure limits the expressive power of the language. However, topologic has the additional modal operator K. As we observed K allows us to express *traveling* within the given open.Thus it enlarges the language just as it was described in [13] semantically. Hence, also as the authors underlined, [5] basicly extends the semantic in [13] and gives a mathematical completeness proof for it.

We also would like to point out the expressive power of topologic. We already observed that, open and closed sets, and dense and nowhere dense sets are expressible in this language. We want to underline the importance of dense sets. It basically tells that we can shrink the given set as much as we want, but we will still have a member satisfy the given sentence. This *existential* statement gives rise to the idea of expressibility of numerous topological properties such as compactness and connectedness which are not expressible in basic modal language.

The very first conclusion we draw is the elegance of using logical framework in topology. This approach tries to unify the both theories in their meta level considering the fact that they both have some certain algebraic properties. But, what this approach lacks is the fact that it cannot cover the whole system of topology with its one modal operator.

On the other hand subset spaces seem to have a slightly more chance to grasp the fundamental notions of point set theory with its two distinct modal operators.

The author of this work looks forward to utilize some more topological (especially geometric topologic) tools two distinguish those two different approaches and learn more about them. As tree structures are often used in (modal) logic, we have the *feeling* that some geometric topological constructions such as *cell-structure* or *genuses* will make the picture more and more clearer and add the notion of *dimension* to the aforementioned approaches.

And who knows, maybe Wittgenstein was wrong.

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