All Normal Extensions of $S5^2$ are Finitely Axiomatizable

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All Normal Extensions of $S^5$ are Finitely Axiomatizable

Road-Map and Recall
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  Recall

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All Normal Extensions of $S_5^2$ are Finitely Axiomatizable

Road-Map and Recall

Road-Map

Road-Map For the Proof

- Recap
  - Necessary mathematical machinery
  - Proof in several steps
  - Complexity results
All Normal Extensions of $S5^2$ are Finitely Axiomatizable

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Road-Map and Recall

Recall

Axioms of $S5^2$

- All tautologies of propositional calculus
- $□_i(φ \to ψ) \to (□_iφ \to □_iψ)$
- $□_iφ \to φ$
- $□_iφ \to □_i□_iφ$
- $◊_i□_iφ \to φ$
- $□_1□_2φ \leftrightarrow □_2□_1φ$

Two modal operators: $□_1$ and $□_2$
Closed under MP and Necessitation (from φ infer $□_iφ$).
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Road-Map and Recall

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- All tautologies of propositional calculus
- $\square_i(\varphi \rightarrow \psi) \rightarrow (\square_i\varphi \rightarrow \square_i\psi)$
- $\square_i\varphi \rightarrow \varphi$
- $\square_i\varphi \rightarrow \square_i\square_i\varphi$
- $\diamondsuit_i\square_i\varphi \rightarrow \varphi$
- $\square_1\square_2\varphi \leftrightarrow \square_2\square_1\varphi$

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Road-Map and Recall

Recall

Facts on $\mathbf{S5}^2$ (1)

Complete with respect to $\{n \times n : n \geq 1\}$, for natural number $n$ [Segerberg].

where we have:

$(x_1, x_2) R_1 (y_1, y_2) \text{ iff } x_2 = y_2$

$(x_1, x_2) R_2 (y_1, y_2) \text{ iff } x_1 = y_1$
Facts on $\mathbf{S5}^2$ (2)

Every proper extension $L$ of $\mathbf{S5}^2$ has poly-size model property; that is, there is a polynomial $P(n)$ such that any $L$-consistent formula $\varphi$ has a model over a frame validating $L$ with at most $P(|\varphi|)$ points, where $|\varphi|$ is the length of the formula $\varphi$. 
Facts on $\text{S}5^2$ (3)

$\mathcal{F} = (W, R_1, R_2)$ is a $\text{S}5^2$ frame where:

- $W$ is non-empty
- $R_i$'s are equivalence relations on $W$ such that

$$\mathcal{F} \models (\forall w, v, u)(wR_1 v \land vR_2 u) \rightarrow (\exists z)(wR_2 z \land zR_1 u)$$
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$p$-morphism

For two $S5^2$ frames $\mathcal{F} = (W, R_1, R_2)$ and $\mathcal{G} = (U, S_1, S_2)$, $p$-morphism $f : U \to W$ from $\mathcal{G}$ to $\mathcal{F}$, for each $i = 1, 2$ is defined as follows:

$$(\forall t \in U)(\forall w \in W)(f(t) R_i w \leftrightarrow (\exists u \in U)(t S_i u \land f(u) = w))$$
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Mathematical Tools

$p$-morphism

Definitions on $p$-morphism

$\textbf{S}5^2$ frames $\mathcal{F}$ is rooted if and only if

$$\forall w, v \exists u (wR_1 u \land uR_2 v)$$

Then define $\textbf{F}_{\textbf{S}5^2}$ as the set of representatives of the isomorphism types of the finite rooted $\textbf{S}5^2$ frames. We will, from now on, consider the frames in $\textbf{F}_{\textbf{S}5^2}$. Why?
Definitions on \( \rho \)-morphism: an earlier result

Let \( L \) be a normal extension of \( S5^2 \). \( \mathcal{F} \in S5^2 \) is called \( L \)-frame if \( \mathcal{F} \) validates each formula in \( L \). Then, define \( F_L \) the set of all \( L \)-frames in \( F_{S5^2} \).

Bezhanisvili proved somewhere else that: \( L \) is complete wrt \( F_L \).

This is the reason why we will only consider the frames in \( F_{S5^2} \). This is the first step towards our aim.

Define \( M_L = \min(F_{S5^2} \setminus F_L) \).
Definitions on $p$-morphism: a relation

We will introduce our first partial order in $\mathbf{FS}_{\mathbf{S}^5^2}$: $\leq$. For $\mathcal{F}$ and $\mathcal{G}$ in $\mathbf{FS}_{\mathbf{S}^5^2}$,

$$\mathcal{F} \leq \mathcal{G} \text{ iff } \mathcal{F} \text{ is a } p\text{-morphc image of } \mathcal{G}.$$ 

For each $\mathcal{G}$ in a subset $A$ of $\mathbf{FS}_{\mathbf{S}^5^2}$, there is a frame $\mathcal{F} \in \text{min}(A)$ such that $\mathcal{F} \leq \mathcal{G}$. 
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— Mathematical Tools
— How to proceed?

Road-Map for the Proof

We will proceed as follows:

- Find a set of formulas that axiomatize any proper normal extension of $\textbf{S}5^2$.
- Show that this set is finite by stating equivalent statement about the finiteness of the set of axioms.
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**Proof**

**Axiomatizability**

**Jankov-Fine Formulas**

\[
\alpha(\mathcal{F}) = \Box_1 \Box_2 \left( \bigvee_{p \in W} (p \wedge \neg \bigvee_{p' \in W \setminus \{p\}} p') \right)
\]

\[
\wedge \bigwedge_{i=1,2} (p \rightarrow \Diamond_i p')
\]

\[
\wedge \bigwedge_{i=1,2} (p \rightarrow \neg \Diamond_i p')
\]

\[
\chi(\mathcal{F}) = \neg \alpha(\mathcal{F})
\]
All Normal Extensions of $\mathbf{S5}^2$ are Finitely Axiomatizable

Proof

Axiomatizability

Why on earth do we need that formula?

$\mathcal{F} \leq \mathcal{G}$ if and only if $\mathcal{G} \not\models \chi(\mathcal{F})$.

$\mathcal{G} \in \mathbf{F}_L$ if and only if for no $\mathcal{F} \in \mathbf{M}_L$, $\mathcal{F} \leq \mathcal{G}$, where $\mathbf{M}_L = \min(\mathbf{F}_{\mathbf{S5}^2} \setminus \mathbf{F}_L)$.

**Theorem** Every proper normal extension $L$ of $\mathbf{S5}^2$ is axiomatizable by the axioms of $\mathbf{S5}^2$ and $\{\chi(\mathcal{F}) : \mathcal{F} \in \mathbf{M}_L\}$.

Need to show $\mathbf{M}_L$ is finite!
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Axiomatizability

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Proof

BQO-Theory comes in

A qo-set (1)

$R_i$ – **Depth** of $\mathcal{F}$ is the number of $R_i$-equivalence classes of $\mathcal{F}$. Denote $d_i(\mathcal{F})$.

$n(L)$ is the least $n$ such that, $n \times n \notin F_L$.

- If $\mathcal{F} \in F_L$, then $d_1(\mathcal{F}) < n(L)$ or $d_2(\mathcal{F}) < n(L)$.
- In contrast, if $\mathcal{F}$ is not in $F_L$, i.e. $\mathcal{F} \in M_L$; then $d_1(\mathcal{F}) \leq n(L)$ or $d_2(\mathcal{F}) \leq n(L)$.
- The previous two results give rise to the following fact: $M_L$ is finite iff $\{\mathcal{F} \in M_L : d_i(\mathcal{F}) = k\}$ is finite for each $k \leq n(L)$ where $i = 1, 2$. 
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▶ The previous two results give rise to the following fact: $\mathcal{M}_L$ is finite iff $\{\mathcal{F} \in \mathcal{M}_L : d_i(\mathcal{F}) = k\}$ is finite for each $k \leq n(L)$ where $i = 1, 2$. 
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\[ \text{Proof} \]

\[ \text{BQO-Theory comes in} \]

\section*{A qo-set (2)}

So to prove the finiteness of $M_L$, prove the finiteness of 
\( \{ F \in M_L : d_i(F) = k \} \) for each $k$ while $i = 1, 2$.

But, since $M_L$ is a $\leq$-antichain in $F_{S5^2}$, \textit{instead} show $M_L$ does \textit{not} contain an infinite $\leq$-antichain.
A qo-set (3): A Newer Relation

Fix $k$. WLOG, let $i = 2$. Let $\mathcal{M}_n$ be the set of $n \times k$ matrices $(m_{ij})$ and $\mathcal{M}$ is the collection $\bigcup_{n \in \omega} \mathcal{M}_n$.

$(m_{ij}) \preceq (m'_{ij})$ holds if $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$ and $n \leq n'$ and there is a surjection $f : n' \to n$ such that $m_{f(i)j} \leq m'_{ij}$.

Observe that $(\mathcal{M}, \preceq)$ is a qo-set.
All Normal Extensions of $S5^2$ are Finitely Axiomatizable

Proof

BQO-Theory comes in

A qo-set (4)

Define $H : F^k_{S5^2} \to \mathcal{M}$ by $H(\mathcal{F}) = (m_{ij})$, if $|F_i \cap F^j| = m_{ij}$. $H$ is an order-reflecting injection, where $F^k_{S5^2}$ is the set of frames in $F_{S5^2}$ with $R_2$-depth $k$, $F_i$ is the $i^{th}$ equivalence class of $R_1$ and $F^j$ is the $j^{th}$ equivalence class of $R_2$.

Therefore, for each $\leq$-antichain $\Delta$ in $F^k_{S5^2}$, then $H(\Delta)$ is a $\leq$-antichain.

So, instead, we will show there is no infinite $\leq$-antichains in $\mathcal{M}$. But, instead of dealing with $\leq$, we will define new a quasi-order: $\sqsubset$. 
All Normal Extensions of $S5^2$ are Finitely Axiomatizable

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**Proof**

BQO-Theory comes in

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**A qo-set (5): The Newest Relation**

For $(m_{ij}) \in M_n$ and $(m'_{ij}) \in M_{n'}$:

$(m_{ij}) \sqsubseteq_1 (m'_{ij})$ if there is an injective order-preserving map $\varphi : n \to n'$ such that $m_{ij} \leq m'_{\varphi(i)j}$ for each $i < n$ and $j < k$.

$(m_{ij}) \sqsubseteq_2 (m'_{ij})$ if there is a map $\psi : n' \to n$ such that $m_{\psi(i)j} \leq m'_{ij}$ for each $i < n$ and $j < k$.

$\sqsubseteq$ is the intersection of $\sqsubseteq_1$ and $\sqsubseteq_2$.

Thus, if $(m_{ij}) \sqsubseteq (m'_{ij})$, then $(m_{ij}) \preceq (m'_{ij})$. 
All Normal Extensions of $S5^2$ are Finitely Axiomatizable

Proof

BQO-Theory comes in

BQOs: finally

Therefore, *instead*, we will show there is no infinite $\sqsubseteq$-antichains in $\mathcal{M}$.

**FACT:** There is no infinite antichains in a BQO.
BQOs: recap

- $(\omega, \leq)$ is a BQO.
- Any suborder of a BQO, and the intersection of two BQOs are BQOs.
- If $(Q, \leq)$ is a BQO, then, $(\wp(Q), \leq)$ is a BQO.
- If $(Q, \leq)$ is a BQO, then $(\bigcup_{\alpha \in On} Q^\alpha, \leq^*)$ is a BQO. Hence, the suborders $(Q^k, \leq^*)$ and $\bigcup_{n<\omega} Q^n, \leq^*$ are BQOs.

Define $\leq^*$ on the class $\bigcup_{\alpha \in On} Q^\alpha$ by $(x_i)_{i<\alpha} \leq^* (y_i)_{i<\beta}$ if there is an order-preserving map $\varphi: \alpha \to \beta$ such that $x_i \leq y_{\varphi(i)}$ for each $i < \alpha$. 
Result-1

- $(\mathcal{M}, \sqsubseteq_1)$ is a BQO.

- $(\mathcal{M}, \sqsubseteq_2)$ is a BQO.

- Thus, $(\mathcal{M}, \sqsubseteq)$ is a BQO.
All Normal Extensions of $S^5_2$ are Finitely Axiomatizable

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► Thus, $(\mathcal{M}, \sqsubseteq)$ is a BQO.
All Normal Extensions of $\mathbf{S5}^2$ are Finitely Axiomatizable

Proof

Results

Result (theorem)

**THEOREM:** All normal extensions of $\mathbf{S5}^2$ are finitely axiomatizable.
All Normal Extensions of $\mathbf{S5}^2$ are Finitely Axiomatizable

Proof

Results

Result (proof)

$\mathbf{S5}^2$ is finitely axiomatizable.

If $L$ is a proper extension of $\mathbf{S5}^2$, then it is axiomatizable by the axioms of $\mathbf{S5}^2$ and \{\(\chi(\mathcal{F}) : \mathcal{F} \in \mathcal{M}_L\}\}.

Since $\sqsubseteq$ is a BQO, it has no $\sqsubseteq$-infinite antichains, and there is no $\preceq$-antichains in $\mathcal{M}$.

Therefore for each $k \in \omega$, $\mathbf{F}_{\mathbf{S5}^2}^k$ has no infinite antichains. Thus, for each $k \leq n(L)$, the set \{\(\mathcal{F} \in \mathcal{M}_L : d_i(\mathcal{F}) = k\)\} has finite number of elements.

Hence, $\mathcal{M}_L$ is finite, and there are only finitely many $\chi(\mathcal{F})$ formulas that axiomatize $L$. 
All Normal Extensions of $\mathbf{S5}^2$ are Finitely Axiomatizable

\[\text{Proof}\]

\[\text{Results}\]

Result (proof)

- $\mathbf{S5}^2$ is finitely axiomatizable.
- If $L$ is a proper extension of $\mathbf{S5}^2$, then it is axiomatizable by the axioms of $\mathbf{S5}^2$ and $\{\chi(\mathcal{F}) : \mathcal{F} \in M_L\}$.
- Since $\sqsubseteq$ is a BQO, it has no $\sqsubseteq$-infinite antichains, and there is no $\preceq$-antichains in $\mathcal{M}$.
- Therefore for each $k \in \omega$, $\mathbf{F}_k^{\mathbf{S5}^2}$ has no infinite antichains. Thus, for each $k \leq n(L)$, the set $\{\mathcal{F} \in M_L : d_i(\mathcal{F}) = k\}$ has finite number of elements.
- Hence, $M_L$ is finite, and there are only finitely many $\chi(\mathcal{F})$ formulas that axiomatize $L$. 
S5^2 is finitely axiomatizable.

If L is a proper extension of S5^2, then it is axiomatizable by the axioms of S5^2 and \( \{ \chi(F) : F \in M_L \} \).

Since \( \sqsubseteq \) is a BQO, it has no \( \sqsubseteq \)-infinite antichains, and there is no \( \preceq \)-antichains in \( M \).

Therefore for each \( k \in \omega, F_{S5^2}^k \) has no infinite antichains. Thus, for each \( k \leq n(L) \), the set \( \{ F \in M_L : d_i(F) = k \} \) has finite number of elements.

Hence, \( M_L \) is finite, and there are only finitely many \( \chi(F) \) formulas that axiomatize \( L \).
Result (proof)

- **S5^2** is finitely axiomatizable.
- If $L$ is a *proper* extension of $S5^2$, then it is axiomatizable by the axioms of $S5^2$ and $\{\chi(\mathcal{F}) : \mathcal{F} \in M_L\}$.
- Since $\sqsubseteq$ is a BQO, it has no $\sqsubseteq$-infinite antichains, and there is no $\preceq$-antichains in $M$.
- Therefore for each $k \in \omega$, $F^k_{S5^2}$ has no infinite antichains. Thus, for each $k \leq n(L)$, the set $\{\mathcal{F} \in M_L : d_i(\mathcal{F}) = k\}$ has finite number of elements.
- Hence, $M_L$ is finite, and there are only finitely many $\chi(\mathcal{F})$ formulas that axiomatize $L$. 

All Normal Extensions of $S5^2$ are Finitely Axiomatizable
Result (proof)

- $\mathbf{S}^2$ is finitely axiomatizable.
- If $L$ is a proper extension of $\mathbf{S}^2$, then it is axiomatizable by the axioms of $\mathbf{S}^2$ and $\{\chi(\mathcal{F}) : \mathcal{F} \in \mathcal{M}_L\}$.
- Since $\sqsubseteq$ is a BQO, it has no $\sqsubseteq$-infinite antichains, and there is no $\preceq$-antichains in $\mathcal{M}$.
- Therefore for each $k \in \omega$, $F^k_{\mathbf{S}^2}$ has no infinite antichains. Thus, for each $k \leq n(L)$, the set $\{\mathcal{F} \in \mathcal{M}_L : d_i(\mathcal{F}) = k\}$ has finite number of elements.
- Hence, $\mathcal{M}_L$ is finite, and there are only finitely many $\chi(\mathcal{F})$ formulas that axiomatize $L$. 
All Normal Extensions of $S5^2$ are Finitely Axiomatizable

- Complexity Results
- Some Facts

**SAT**

- $S5^2$ has a exponential size model property, and its satisfiability problem is NEXP-TIME.
- Every proper normal extension of $S5^2$ is decidable in polynomial time. Therefore, together with the poly-size model property, it implies that the satisfiability for the normal proper extension is NP-complete.

**Poly-size model property** For the each proper normal extension $L$ of $S5^2$, there is a polynomial $P(n)$ s.t. for any $L$-consistent formula $\phi$ has a model over a frame validating $L$, and model has at most $P(|\phi|)$ points where $P(|\phi|)$denotes the length of $\phi$. 
SAT

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**Poly-size Model Property** For the each proper normal extension \( L \) of \( S5^2 \), there is a polynomial \( P(n) \) s.t. for any \( L \)-consistent formula \( \phi \) has a model over a frame validating \( L \), and model has at most \( P(|\phi|) \) points where \( P(|\phi|) \)denotes the length of \( \phi \).
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Thanks for your attention