Insensitive Games: Game Semantics for Modal Insensitivity

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Abstract. In this paper, we introduce a game theoretical semantics for a reflexive insensitive logic, and observe how classical semantic games needs to be altered to allocate reflexive insensitivity. Following, we extend semantic games to develop new games and new modalities. We prove the correctness theorems in each case.

Keywords: Game semantics · Reflexive Insensitivity · Semantic Insensitivity · Non-normal Modal Logics

1 Introduction

Insensitivity phenomenon in modal logic is the indifference of truth to the existence or non-existence of certain semantic properties. Reflexive-insensitivity is a well-known case where the logic is insensitive to the existence or non-existence of reflexive arrows in the relational model [12]. They were initially introduced to answer certain formal epistemic problems in modal logic such as contingency, essence and accident, and ignorance [11,13,15,17,18]. Recently, there has been a topological approach to the subject as well [5].

The idea of insensitivity is interesting from a game theoretical angle, too. Existence or non-existence of certain relational arrows describe the possibility or impossibility of certain moves in a game, which in turn impacts players' strategies. The game theoretical limitations that logics of insensitivities introduce to semantic games make such games of interest. Conversely, from a logical view point, semantic games offer an alternative semantics for such logical systems where the truth of a formula in a given model is established by playing a game. As such, semantic games introduce the element of strategising, thus rationality and choices, to logical discourse. An interesting research programme in this area is to develop semantic games for non-classical logics, where the non-classical elements in such logics generate rather interesting and *non-classical* games [1,2,6].

A proof theoretical analysis of the insensitivity phenomenon was discussed in [19]. The work focused on tableaux methods, reinforcing the analytical

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approaches to insensitivity. This is particularly important considering the BHK connection between proofs and truth, where to be true means to have a proof. Game semantics follows a similar pattern.

We start by introducing the formal system for reflexive-insensitivity. Let $\mathbf{P}$ be a set of countable propositional variables. We define the language of reflexive-insensitive modal logic $\mathcal{L}_\circ$ as follows in the Backus–Naur form

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \circ\varphi \mid \bullet\varphi$$

where $p \in \mathbf{P}$.

The language of basic modal logic $\mathcal{L}_\Box$ is given as follows.

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi$$

where, similarly, $p \in \mathbf{P}$.

We take the conditional arrow as an abbreviation: $\varphi \to \psi \equiv \neg\varphi \vee \psi$. Similarly, $\Diamond\varphi \equiv \neg\Box\neg\varphi$, dually. We prefer to keep both $\Diamond$ and $\Box$ modalities for game theoretical completeness of our treatment. The basic language of propositional logic, without the modal operators, will be denoted by $\mathcal{L}$.

A model $M = (W, R, V)$ is a triple where W is a non-empty set of worlds, R is a binary relation $R \subseteq W^2$ defined on W, and $V : W \mapsto 2^{\mathbf{P}}$ is a valuation function mapping worlds to sets of propositions that are satisfied there. In this model, the semantics for the Booleans are straight-forward. The semantics for the classical modalities and the $\circ$ and $\bullet$ operators are given as follows.

$M, w \models \Box\varphi$ iff $\forall v.wRv$ implies $M, v \models \varphi$
$M, w \models \Diamond\varphi$ iff $\exists v.wRv$ such that $M, v \models \varphi$
$M, w \models \circ\varphi$ iff $M, w \not\models \varphi$, or $\forall v.wRv$ implies $M, v \models \varphi$
$M, w \models \bullet\varphi$ iff $M, w \models \varphi$, and $\exists v.wRv$ such that $M, v \not\models \varphi$

Using the language of basic modal logic, we then observe the following:

$$\circ\varphi \equiv \neg\varphi \vee \Box\varphi$$
$$\bullet\varphi \equiv \varphi \wedge \Diamond\neg\varphi$$

This paper is organised as follows. After briefly introducing the game semantics for classical modal logics, we develop a semantic game for reflexive-insensitive logics. After proving the correctness of semantic games, we then discuss how such games can be generalised to a broader class of logics of insensitivities.

2 Game Semantics for Classical Modal Logics, Briefly

Let $M = (S, R, V)$ be a model as before, and F be a frame based on M defined as $F = (S, R)$. Given a model, we construct modal semantic games.

Definition 1. *A semantic game is a tuple $\Gamma = (\pi, \rho, \sigma, F, w)$ where*

- *π is the set of players (with cardinality 2 in classical cases),*

– *ρ is the set of well-defined game rules,*
– *σ is the set of positions,*
– *F is the modal frame on which the game is played,*
– *w is the state in set S where the game starts.*

In classical modal logics, we have two players in π: the Verifier and the Falsifier. The Verifier's goal is to force the game to a win with true propositions whereas the Falsifier's goal is to force the game to a win with false propositions.

The positions σ depend on subformulas, players and states, and are determined by the game rules. The set σ is composed of tuples of the form (π_i, φ, w) where $\pi_i \in \pi$ is a player, $\varphi \in \mathcal{L}_\Box$ is a well-defined formula and $w \in S$ is a state. As such, the position will include the "turn function" which indicates which players are supposed to make a move at each position and at each state.

The rules ρ are defined inductively as transformations from a game position (π_i, φ, w) to a set of game positions $\{(\pi_j, \psi, v)\}_{j \in I}$ for $\pi_i, \pi_j \in \pi$, $I \subseteq \pi$ and $v \in S$. The players, formulas and the states in game positions are determined by the game rules and the relation R.

The set of accessible states from w is denoted by $R[w]$, and defined as usual: $R[w] := \{v : R(w, v)\}$.

Semantic games for classical modal logics are defined as follows.

Definition 2. *The tuple $\Gamma_\Box = (\varphi, \pi, \rho, \sigma, F, w)$ is a semantic game for classical modal logic with the language $\mathcal{L}_\Box$ for the formula $\varphi \in \mathcal{L}_\Box$, where $\pi = \{Verifier, Falsifier\}$, σ is the set of tuples (π_i, φ, w) for $\pi_i \in \pi$, $\varphi \in \mathcal{L}_\Box$ and $w \in S$, $F = (S, R)$ is the modal frame on which the game is played, and $w \in S$ is a state where the game starts.*

We define the set of positions σ inductively as follows, where $w \in S$:

(σ_p) If φ is atomic, then $(\pi_i, \varphi, w) \in \sigma$ for all $\pi_i \in \pi$,
($\sigma_\neg$) If $\varphi = \neg\psi$, then $(\pi_i, \varphi, w) \in \sigma$ for all $\pi_i \in \pi$, and $(\pi_j, \psi, w) \in \sigma$ for some $\pi_j \in \pi$ depending on ψ's main connective;
($\sigma_\wedge$) If $\varphi = \psi \wedge \chi$, then (Falsifier, $\varphi, w) \in \sigma$, and $(\pi_j, \psi, w), (\pi_k, \chi, w) \in \sigma$ for some $\pi_j, \pi_k \in \pi$ depending on ψ and χ's main connectives;
($\sigma_\vee$) If $\varphi = \psi \vee \chi$, then (Verifier, $\varphi, w) \in \sigma$, and $(\pi_j, \psi, w), (\pi_k, \chi, w) \in \sigma$ for some $\pi_j, \pi_k \in \pi$ depending on ψ and χ's main connectives;
($\sigma_\Box$) If $\varphi = \Box\psi$, then (Verifier, $\varphi, w) \in \sigma$, and $(\pi_j, \psi, v) \in \sigma$ for some $\pi_j \in \pi$ depending on ψ's main connective, and for all $v \in R[w]$;
($\sigma_\Diamond$) If $\varphi = \Diamond\psi$, then (Falsifier, $\varphi) \in \sigma$, and $(\pi_j, \psi, v) \in \sigma$ for some $\pi_j \in \pi$ depending on ψ's main connective, and for all $v \in R[w]$.

The set of rules ρ is defined as follows:

(ρ_p) If φ is atomic at w, the game terminates at w; and the Verifier wins if φ is true, the Falsifier wins if φ is false;
($\rho_\neg$) If $\varphi = \neg\psi$ at w, the Falsifier and the Verifier switch roles, and the game continues with ψ at w;

($\rho_\vee$) If $\varphi = \psi \vee \chi$ at w, the Verifier chooses between ψ and χ at w;
($\rho_\wedge$) If $\varphi = \psi \wedge \chi$ at w, the Falsifier chooses between ψ and χ at w;
($\rho_\Box$) If $\varphi = \Box\psi$, the Falsifier chooses a state $v \in R[w]$, and the game continues with ψ at v;
($\rho_\Diamond$) If $\varphi = \Diamond\psi$, the Verifier chooses a state $v \in R[w]$, and the game continues with ψ at v.

A brief explanation of this definition is in order. Let us consider the case for negation. For $\varphi = \neg\psi$, for any ψ, *all* players (which are the Verifier and the Falsifier, in the case of classical modal logic) make a move. By making that move, the position $(\pi_i, \neg\psi, w)$ is broken down into (π_j, ψ, w). If π_i was the Verifier, then π_j becomes the Falsifier, and *vice versa*. In some non-classical logical games, for example, for negation we can have $\pi_i = \pi_j$ –that is some players may *not* switch roles under negation [1]. Therefore, it is relatively straight-forward to extend Definition 2 to other logics.

Furthermore, the above definition is scaleable. It specifies the players, and more importantly the positions. In chess, for example, positions are given similarly. The position Be5 in coloumn 1, suggests that the player White moves Bishop to square e5, containing the same three components as in the tuples in σ: the player, the state and the proposition. The game rules, however, are given inductively where for each possible position, it specifies which moves are allowed and whose turn it is to make a move. As such, this definition can easily be extended to various other logics where alternative rules can be introduced.

The well-known correctness theorem of game semantics establishes the connection between modal semantic games and (classical) modal logic as follows.

Theorem 1. *Given a modal model $M = (S, R, V)$ and a modal semantic game $\Gamma_\Box = (\varphi, \pi, \rho, \sigma, F, w)$, we have*

$$M, w \models \varphi \text{ if and only if the Verifier has a winning strategy in } \Gamma_\Box \text{ at } w.$$

In classical modal logic, semantic games are two-player, zero-sum, competitive, determined and sequential with perfect information. It is, however, not a necessity to have all the aforementioned properties in a semantic game. Various non-classical logics force semantic games to be, for instance, a three-player, non-zero sum, cooperative, non-determined, non-sequential or concurrent [1–3,6]. Therefore, the relation between semantic games and logic are two directional: Given a logic, one can aim at developing a semantic game for that logic, or given a semantic game with various game theoretical properties, one can aim at engineering a logical system that matches with that game with the said properties. The current paper contributes to this research programme by examining logics of reflexive-insensitivity.

Let us now we examine what reflexive-insensitivity introduces to semantic games by focusing on the logic-to-games direction of the aforementioned relation.

3 Semantic Games for Reflexive-Insensitivity

The idea behind representing reflexive insensitivity game theoretically is to allow the players to have an *indifference* to the existence and the accessibility of some positions in the game board. Those positions will create certain strategic power for some players. Let us start with expanding Definition 2 to define semantic games for reflexive-insensitive logics.

Definition 3. *The tuple $\Gamma_\circ = (\varphi, \pi, \rho, \sigma, F, w)$ is a semantic game for reflexive-insensitive modal logic with the language $\mathcal{L}_\circ$ for the formula $\varphi \in \mathcal{L}_\circ$, where $\pi = \{Verifier, Falsifier\}$, σ is the set of tuples (π_i, φ, w) for $\pi_i \in \pi$, $\varphi \in \mathcal{L}_\Box$, and $w \in S$, $F = (S, R)$ is the modal frame where the game is played on, and $w \in S$ is a state where the game starts.*

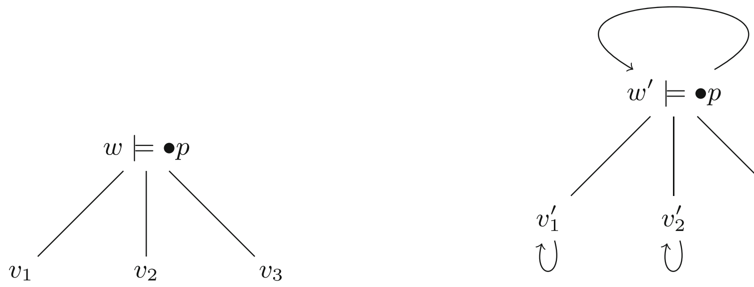
We define the set of positions σ inductively as follows, where $w \in S$:

- *(σ_p) If φ is atomic, then $(\pi_i, \varphi, w) \in \sigma$ for all $\pi_i \in \pi$;*
- *($\sigma_\neg$) If $\varphi = \neg\psi$, then $(\pi_i, \varphi, w) \in \sigma$ for all $\pi_i \in \pi$, and $(\pi_j, \psi, w) \in \sigma$ for some $\pi_j \in \pi$ depending on ψ's main connective;*
- *($\sigma_\wedge$) If $\varphi = \psi \wedge \chi$, then $(Falsifier, \varphi, w) \in \sigma$, and $(\pi_j, \psi, w), (\pi_k, \chi, w) \in \sigma$ for some $\pi_j, \pi_k \in \pi$ depending on ψ and χ's main connectives;*
- *($\sigma_\vee$) If $\varphi = \psi \vee \chi$, then $(Verifier, \varphi, w) \in \sigma$, and $(\pi_j, \psi, w), (\pi_k, \chi, w) \in \sigma$ for some $\pi_j, \pi_k \in \pi$ depending on ψ and χ's main connectives;*
- *($\sigma_\circ$) If $\varphi = \circ\psi$, then $(Verifier, \varphi, w) \in \sigma$, and $(\pi_j, \neg\psi, w) \in \sigma$ and $(\pi_j, \psi, v) \in \sigma$ for some $\pi_j \in \pi$ depending on ψ's main connective, and for all $v \in R[w]$;*
- *($\sigma_\bullet$) If $\varphi = \bullet\psi$, then $(Falsifier, \varphi, w) \in \sigma$, and $(\pi_j, \psi, w) \in \sigma$ $(\pi_j, \psi, v) \in \sigma$ for some $\pi_j \in \pi$ depending on ψ's main connective, and for all $v \in R[w]$.*

The set of rules ρ is defined as follows:

- *(ρ_p) If φ is atomic at w, the game terminates at w; and the Verifier wins if φ is true, the Falsifier wins if φ is false;*
- *($\rho_\neg$) If $\varphi = \neg\psi$ at w, the Falsifier and the Verifier switch roles, and the game continues with ψ at w;*
- *($\rho_\vee$) If $\varphi = \psi \vee \chi$ at w, the Verifier chooses between ψ and χ at w;*
- *($\rho_\wedge$) If $\varphi = \psi \wedge \chi$ at w, the Falsifier chooses between ψ and χ at w;*
- *($\rho_\circ$) If $\varphi = \circ\psi$, the Verifier first chooses between remaining at the current state w and moving onto a state $v \in R[w]$. If he chooses to remain, then the game continues with $\neg\psi$ at w; if he chooses the latter, then the Falsifier makes another choice amongst the states in $R[w]$ where the game continues with ψ at a $v \in R[w]$;*
- *($\rho_\bullet$) If $\varphi = \bullet\psi$, the Falsifier first chooses between remaining at the current state w and moving onto a state $v \in R[w]$. If she chooses the former, the game continues with ψ at w; if she chooses the latter, the Verifier makes another choice amongst the states in $R[w]$ where the game continues with $\neg\psi$ at a $v \in R[w]$.*

The rules ($\rho_\circ$) and ($\rho_\bullet$) are more complicated than their modal counterparts ($\rho_\Box$) and ($\rho_\Diamond$), which were given in Definition 2. The main difference is that the modal operators $\circ$ and $\bullet$ force two sequential-moves with an alternating order of players. One player makes a choice and then "passes the ball" to the other player. Let us examine what they actually entail game theoretically by considering a simple example.



(a) The model M without the reflexive arrows. (b) The model M' with the reflexive arrows.

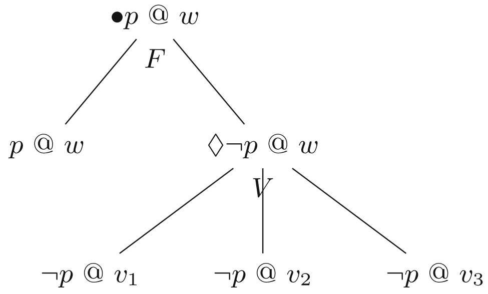
Fig. 1. Models M and M', where the latter has reflexive arrows.

We start with a game for $\bullet p$ at a state w. In order to underline the reflexive-insensitivity of the logic, let us consider the following two models M and M', and evaluate $\bullet p$ at both w in M and w' in M', as given in Fig. 1.

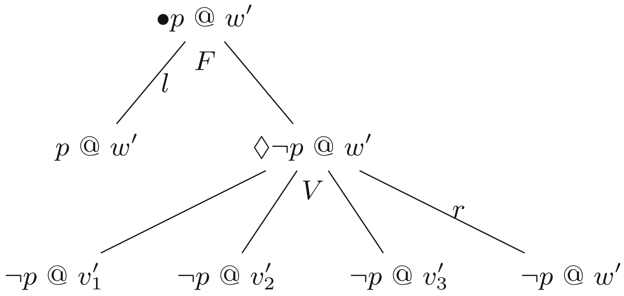
Based on the above models M and M', we next construct game models, respectively at w and w', evaluating the formula $\bullet p$, given in Fig. 2. The player Falsifier is represented by F whereas the Verifier is by V where they get to make moves.

A closer inspection of the game tree given in Fig. 2b for M' shows how the reflexive arrows change the game tree. The moves l and r may seem to create a contradiction at first glance. However, the order of making a choice for the player F makes it impossible to make contradictory choices. Once the move l is made, r cannot be chosen at the same time. Therefore, the contradictory choices cannot be made – even if the players with opposing goals would rationally want to do so. Thus, the game becomes *insensitive* to the additional r moves, which can contradict *previous* moves of the opponent. This is the game theoretical characterisation of the insensitivity phenomenon within the context of semantic games. The strength of this approach is its *scalability* – it can be extended to other semantic modal insensitivities.

Another strength of the game semantic approach to insensitivity is that it resorts to "move priority order" to explain the phenomenon. Once a choice is made by a player, alternative moves are eliminated from the game tree. As a result of this, players can make two sequential moves without a contradiction. This is similar to Iterated Elimination of Strictly Dominated Strategies, which is already introduced into semantic games for the logics of non-sense [2]. However,



(a) The game tree based on M.



(b) The game tree based on M'.

Fig. 2. Game trees based on M and M', where the resolution of negated propositions $\neg p$ are omitted for easy read.

in the context of insensitive logics, it is not the strategies that are eliminated but redundant moves to which the game is insensitive. As such, semantic games for reflexive insensitivities extend the way sequentiality works in classical semantic games. Therefore, the non-classical element in such games is restricting the way that sequentiality of players' moves works.

A strategic power of insensitivity within the context of semantic games is worth noting. With the modalities $\circ$ and $\bullet$, players have the strategic power of *controlling the turns* in the game. They may choose to make a move themselves, or "pass the ball" to the opponent. This changes the strategic dynamic of the game that is familiar from the game semantics of standard modalities, given in rules $(\rho_\Box)$ and $(\rho_\Diamond)$ in Definition 2. Furthermore, this idea can be generalised for the sake of the games-to-logic direction. One can imagine a game where the players have certain power to control the turns, where a player's move may entail that first F then V, and then F again would make moves sequentially, following the said order. The logic that can match this strategic power may have some insensitivity to certain modal formulas. We discuss this idea in due time.[1]

[1] It is important to note that there is a sense of *completeness* in this bidirectional relation between logic and games from the insensitivity phenomena. The question of whether there exists a logic for every finite sequence of turns falls outside the scope of this paper.

Now, we test the ideas which we discussed earlier with a semantic game for the formula $\circ\varphi$ at w in M' with a slight abuse of the language for clarity. The game tree given in Fig. 3 illustrates the moves generated by reflexive arrows. The game remains strategically insensitive to the moves l and r.

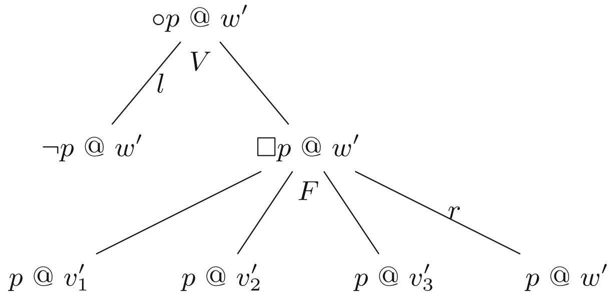


Fig. 3. Game tree for the semantic game for the formula $\circ\varphi$ at w' in model M'.

We now conclude this section with the following correctness theorem.

Theorem 2. *Given a reflexive-insensitive model $M = (S, R, V)$ and a reflexive-insensitive semantic game $\Gamma_\circ = (\varphi, \pi, \rho, \sigma, F, w)$, we have*

$M, w \models \varphi$ *if and only if the Verifier has a winning strategy for φ in $\Gamma_\circ$ at w.*

Similarly,

$M, w \not\models \varphi$ *if and only if the Falsifier has a winning strategy for φ in $\Gamma_\circ$ at w.*

4 Generalised Insensitive Games

An interesting dimension of game semantics for non-classical logics is that one can take advantage of game theoretical tools to develop new logics.

For instance, in the case of the logics of non-sense of Bochvar-Halldén, it is possible to extend the game semantics of such systems to engineer various extensions [2,7–10,14]. In the semantic games for Bochvar-Halldén logics, winning strategies of players form a partial order of *dominance*, reminiscent of the game theoretical method of *iterated elimination of strictly dominated strategies* (IESDS).

IESDS makes it possible to eliminate those strategies which are strictly dominated by other strategies. Carrying out this method step by step, one reaches a solution [16]. In logics of non-sense, this establishes truth and allows one to extend the standard 3-valued logic of non-sense of Bochvar-Halldén to a chain of n-valued logics of non-sense [2]. A similar method can also be a applied to diagrammatic reasoning in logic whilst establishing a connection between game semantics and diagrammatic reasoning in logic [3].

In the case of reflexive-insensitive logics, it is possible to develop an extension of $\Gamma_\circ$ games. The curious question is what the logics that correspond to these games would be. This is our goal in this section.

In $\Gamma_\circ$ games, moves in the form of "I play, and then you play" are allowed. This is a restriction in game semantics only because the shape of formulas determine the order of play. An immediate extension of $\Gamma_\circ$ would be to engineer games which would allow moves in the form of "I play, you play, and then I play again" for both players. This approach could also enable us to form a chain of games where it is possible to have games with moves "I play, you play, I play and then you play", and so on. Simply put, we already know the logic for semantic games with moves of the form "I play, and then you play", then what is the logic for semantic games which allow moves of the form "I play, you play, and then I play again"? That is what we answer in sequel.

We start with proposing the game rules with such moves. Let us denote such modalities with $\triangle$ and $\blacktriangle$. The game rules for the new modalities are given as follows.

($\rho_\triangle$) If $\varphi = \triangle\psi$, the Verifier first chooses between remaining at the current state w and moving onto a state $v \in R[w]$. If he chooses to remain, then the game continues with $\neg\psi$ at w; if he chooses the latter, then the Falsifier first chooses between remaining at state v in $R[w]$ and moving onto a state $u \in R[v]$. If the Falsifier chooses to remain at v, then the game continues with $\neg\psi$ at v. If the Falsifier chooses to move onto $t \in R[v]$, then the Verifier chooses a $t \in R[v]$ and the game continues with ψ at t.

($\rho_\blacktriangle$) If $\varphi = \blacktriangle\psi$, the Falsifier first chooses between remaining at the current state w and moving onto a state $v \in R[w]$. If he chooses to remain, then the game continues with ψ at w; if he chooses the latter, then the Verifier first chooses between remaining at state v in $R[w]$ and moving onto a state $u \in R[v]$. If the Verifier chooses to remain at v, then the game continues with *psi* at v. If the Verifier chooses to move onto $t \in R[v]$, then the Falsifier chooses a $t \in R[v]$ and the game continues with $\neg\psi$ at t.

The game trees for the formulas $\triangle p$ and $\blacktriangle p$ at state w are given as follows in Fig. 4, where $v \in R[w]$ and $u \in R[v]$.

It is possible to iterate the game rules to create a rule which is of the form "I play, you play, I play, then you play again". Let us briefly illustrate this case, too. For the four-turn moves, the corresponding modalities will be denoted by $\triangledown$ and $\blacktriangledown$. Consequently, let us call these moves ($\rho_\triangledown$) and ($\rho_\blacktriangledown$), and give them as follows.

($\rho_\triangledown$) If $\varphi = \triangledown\psi$, the Verifier first chooses between remaining at the current state w and moving onto a state $v \in R[w]$. If he chooses to remain, then the game continues with $\neg\psi$ at w; if he chooses the latter, then the Falsifier first chooses between remaining at state v in $R[w]$ and moving onto a state $t \in R[v]$. If the Falsifier chooses to remain at v, then the game continues with $\neg\psi$ at v. If the Falsifier chooses to move onto $t \in R[v]$, then the Verifier

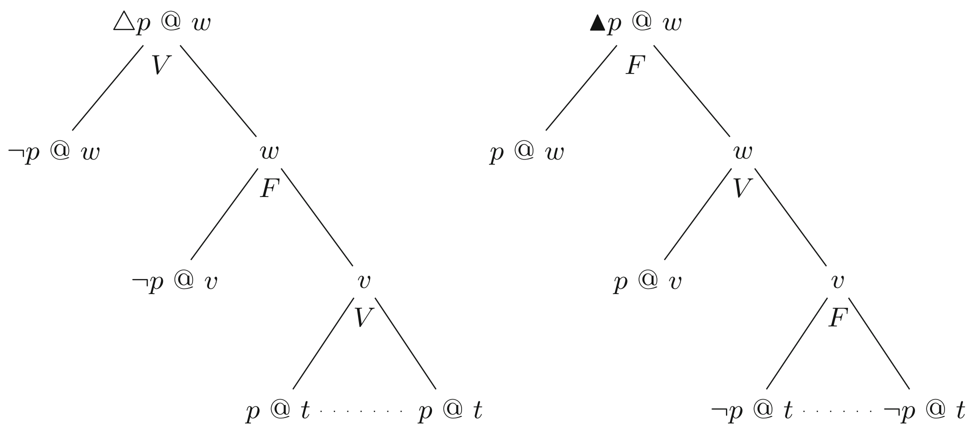


Fig. 4. The game trees for the formulas $\triangle p$ and $\blacktriangle p$ at state w, respectively, where $v \in R[w]$ and $t \in R[v]$.

at t first chooses between remaining at t and moving onto a state $u \in R[t]$. If the Verifier chooses to remain at t, then the game continues with $\neg\psi$ at t. If the Verifier chooses to move onto $u \in R[t]$, then the game continues with ψ at u.

($\rho_{\blacktriangledown}$) If $\varphi = \blacktriangledown\psi$, the Falsifier first chooses between remaining at the current state w and moving onto a state $v \in R[w]$. If he chooses to remain, then the game continues with ψ at w; if he chooses the latter, then the Verifier first chooses between remaining at state v in $R[w]$ and moving onto a state $t \in R[v]$. If the Verifier chooses to remain at v, then the game continues with psi at v. If the Verifier chooses to move onto $t \in R[v]$, then the Falsifier at t first chooses between remaining at t and moving onto a state $u \in R[t]$. If the Falsifier chooses to remain at t, then the game continues with ψ at t. If the Falsifier chooses to move onto $u \in R[t]$, then the game continues with $\neg\psi$ at u.

The game trees for the formulas $\triangledown p$ and $\blacktriangledown p$ at state w, respectively, where $v \in R[w]$, $u \in R[v]$, $t \in R[u]$, are given in Fig. 5.

Let us introduce the $\triangle, \blacktriangle$ and $\triangledown, \blacktriangledown$ operators to the basic language $\mathcal{L}$ which correspond to the game rules $(\rho_{\triangle})$, $(\rho_{\blacktriangle})$, $(\rho_{\triangledown})$ and $(\rho_{\blacktriangledown})$, and denote the language with these four modalities by $\mathcal{L}_{\triangle}$. Similarly, we denote the semantic games played in $\mathcal{L}_{\triangle}$ by $\Gamma_{\triangle}$, and define it in the usual way by extending Definition 2 by rules $(\rho_{\triangle})$, $(\rho_{\blacktriangle})$, $(\rho_{\triangledown})$ and $(\rho_{\blacktriangledown})$.

Now, what is the model theoretical semantics of $\triangle, \blacktriangle$ and $\triangledown, \blacktriangledown$ operators? We propose the following.

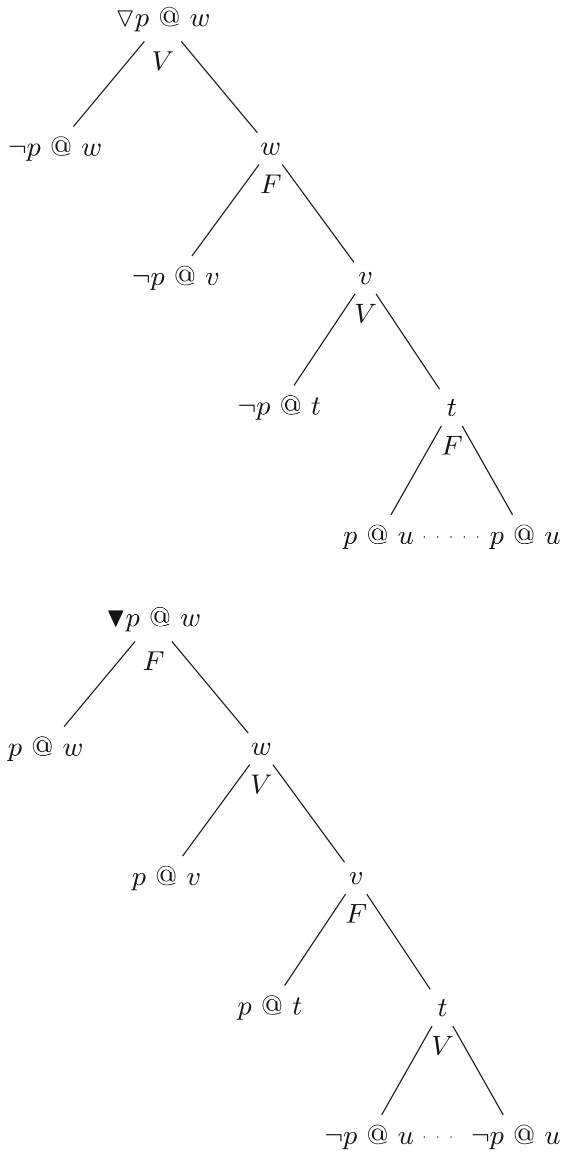


Fig. 5. The game trees for the formulas $\triangledown p$ and $\blacktriangledown p$ at state w, respectively, where $v \in R[w]$, $t \in R[v]$, $u \in R[t]$.

$$\begin{array}{lll}
M, w \models \triangle\varphi & \text{iff} & M, w \not\models \varphi, \text{ or} \\
 & & \forall v.wRv \text{ implies } (v \not\models \varphi \text{ and } (\exists t.vRt \text{ such that } t \models \varphi)) \\
M, w \models \blacktriangle\varphi & \text{iff} & M, w \models \varphi, \text{ and} \\
 & & \exists v.wRv \text{ such that } (v \models \varphi \text{ implies } (\forall t.vRt \text{ implies } t \not\models \varphi))
\end{array}$$

$$\begin{array}{lll}
M, w \models \triangledown\varphi & \text{iff} & M, w \not\models \varphi, \text{ or} \\
 & & \forall v.wRv \text{ implies } (v \not\models \varphi \text{ and} \\
 & & \quad (\exists t.vRt \text{ such that } (t \not\models \varphi \text{ or } (\forall u.tRu \text{ implies } u \models \varphi)))) \\
M, w \models \blacktriangledown\varphi & \text{iff} & M, w \models \varphi, \text{ and} \\
 & & \exists v.wRv \text{ such that } (v \models \varphi \text{ implies} \\
 & & \quad (\forall t.vRt \text{ implies } (t \models \varphi \text{ and } (\exists u.tRu \text{ and } u \not\models \varphi))))
\end{array}$$

We leave it to future work to examine the expressivity of the above modalities in $\mathcal{L}_{\Box}$ and $\mathcal{L}_{\circ}$ as well as the possibility to engineer various combinations of sequences of moves amongst the players.

What we demonstrated in this section is a methodology to generate logics based on various game theoretical *alterations* in semantic games. We took one simple step to form an alternating chain of orders, and it is very well possible to consider the cases that restrict the move orders to "I play, then I play again, then you play and then you play again" and so forth allowing various other combinations of strings of move orders. Such generalisations and their mathematical structures are left for future work.

We conclude this section with the following correctness theorem.

Theorem 3. *Given* $M = (S, R, V)$ *and a semantic game* $\Gamma_{\triangle} = (\varphi, \pi, \rho, \sigma, F, w)$*, we have*

$M, w \models \varphi$ *if and only if the Verifier has a winning strategy for* φ *in* $\Gamma_{\triangle}$ *at* w*.*

Similarly,

$M, w \not\models \varphi$ *if and only if the Falsifier has a winning strategy for* φ *in* $\Gamma_{\triangle}$ *at* w*.*

5 Conclusion

Various game theoretical conditions have a matching idea in logic. For example, the well-known solution method, "the iterated elimination of strictly dominated strategies", is captured by *strategy dominance* [2]. The current paper's contribution can be viewed within the remits of the same research programme.

In this paper, however, we used a much simpler idea from games: "pass the ball". Generalisation of this idea is straight-forward game theoretically with the help of some combinatorics. One can easily imagine a game of n-players, where some moves (that is modalities) enforce a player to pass the ball in a certain way between m-players. This would also help us to *engineer* a logical system with certain insensitivities. Similarly, various combinations of booleans and quantifiers, matching with a different order of players making moves in a certain order, can easily be motivated game theoretically. What is left is to explore the relationship between such sequences of moves and the insensitivity phenomenon. In this paper, particularly, this is how we captured reflexive-insensitivity game theoretically.

Finally, game semantics is helpful to present a diagrammatic reasoning for insensitive logics [3,4]. A visual representation of insensitivity is helpful to expand the strategic and game theoretical analysis to a broader class of insensitivities. What is next is to explore the relation between game trees for various semantic insensitivity and backward induction –a well-known method in game theory– from a diagrammatic view point.

We leave such ideas for future work.

A Appendix for Proofs

Proof (Theorem 2). The proof is by induction on the complexity of formulas. We skip the Boolean cases.

Let $M = (S, R, V)$ be a reflexive-insensitive model and $\Gamma_\circ$ be a reflexive-insensitive semantic game as defined earlier. We start with the case for $\varphi \equiv \circ\psi$.

Suppose $M, w \models \circ\psi$ for $w \in S$. Then, by the semantics of the $\circ$ modality, we have $M, w \not\models \psi$, or $\forall v.wRv$ implies $M, v \models \psi$.

Now, we have two cases to analyse. First, by the induction hypothesis for the Falsifier (applied to $M, w \not\models \psi$), we conclude that the Falsifier has a winning strategy at w for ψ. Second, by the induction hypothesis for the Verifier (applied to $M, v \models \psi$), we conclude that the Verifier has a winning strategy at v for all v with wRv. Furthermore, by the induction hypothesis for the Verifier for disjunction (applied to the main disjunction in the semantics of the $\circ$ modality), we conclude that the Verifier makes the first choice. Putting all together, we conclude that the Verifier first chooses either

(i) to stay at w and game continues with $\neg\psi$ at w, or

(ii) to move to a v in $R[w]$ so that the game continues with ψ at v in $R[w]$.

In the first case, since the Falsifier has a winning strategy for ψ at w, the Verifier has a winning strategy for $\neg\psi$ at w, by the case for the negation of this theorem. In the second case, as the Verifier is making the choice for $v \in R[w]$ and by the induction hypothesis for ψ at v, we conclude that the Verifier is still has a winning strategy at w. The very choice between these two cases is made by the Verifier, so either case, the Verifier has a winning strategy at w for the formula $\circ\psi$, by the game rule $\rho_\circ$ and the induction hypothesis. This concludes the truth-to-strategy direction of the proof.

For the converse direction, from strategy-to-truth, let us first suppose that the Verifier has a winning strategy in $\Gamma_\circ$ at w for the semantic game for the formula $\circ\psi$. By the game rule $(\rho_\circ)$ for $\circ\psi$ in $\Gamma_\circ$, the Verifier gets to make a move. He will choose either

(i) to remain at w so that the game continues with $\neg\psi$ at w, or

(ii) to move to $v \in R[w]$ so that the game continues with ψ at v.

In the first case, by the induction hypothesis of the very theorem, we conclude that $M, w \not\models \psi$. In the second case, using the same reasoning, we conclude that $M, v \models \psi$ for all $v \in R[w]$. Moreover, as the very first choice between the cases (i) and (ii) was carried out by the Verifier, by the induction hypothesis for the case for disjunction, we finally deduce that $M, w \not\models \psi$ or $M, v \models \psi$ for all $v \in R[w]$. Finally, by the semantics of the $\circ$ modality, we have $M, w \models \circ\psi$. This concludes the strategy-to-truth direction of the proof.

The argument for the Falsifier and the negation is almost identical.

The case for $\varphi \equiv \bullet\psi$ is similar, hence skipped.

This completes the proof.

Proof (Theorem 3). The proof is by induction on the complexity of formulas. We skip the Boolean cases. Therefore, we only prove the theorem for $\varphi = \triangle\psi$.

Let $M = (S, R, V)$ be a model and $\Gamma_\triangle$ be a semantic game as defined earlier. We start with the case for $\varphi \equiv \triangle\psi$.

Suppose $M, w \models \triangle\psi$ for $w \in S$. Then, by the semantics of the $\triangle$ modality, we have $M, w \not\models \psi$ or, $\forall v.wRv$ implies ($v \not\models \psi$ and ($\exists t.vRt$ such that $t \models \psi$)).

Now, we have two cases to analyse: cases (1) and (2). First, for the case (1), by the induction hypothesis for the Falsifier (applied to $M, w \not\models \psi$), we conclude that the Falsifier has a winning strategy for ψ at w. The second case is more complicated with two further sub-cases 2(a) and 2(b). First, for the case 2(a), by the induction hypothesis (applied to $v \not\models \psi$), Falsifier has a winning strategy at any state v in $R[w]$. Second, for the case 2(b), by the induction hypothesis (applied to $t \models \psi$) The Verifier has a winning strategy for some state t in $R[v]$. The very choice between the cases 2(a) and 2(b) is made by the Falsifier. And the choice between cases (1) and (2) is made by the Verifier, by the induction hypothesis for disjunction. Therefore, by the game rule $\rho_\triangle$ and the induction hypothesis, the Verifier has a winning strategy for $\triangle\psi$. This was truth-to-strategies direction.

For the converse, the strategies-to-truth direction, assume that the Verifier has a winning strategy in $\Gamma_\triangle$ at w for the semantic game for the formula $\triangle\psi$. Then, by the game rule $\rho_\triangle$, the Verifier makes the first choice. The Verifier chooses either

(Case 1): to remain at w so that the game continues with $\neg\psi$ at w, or

(Case 2): to move to $v \in R[w]$ where there the Falsifier chooses either

(Case 2a): to remain at v so that the game continues with $\neg\psi$ at v, or

(Case 2b): to move to $t \in R[v]$ where the Verifier chooses a $t \in R[v]$ and the game continues with ψ at t.

In Case 1, by the induction hypothesis of the very theorem, we conclude that $M, w \not\models \psi$. This is a choice made by the Verifier.

In Case 2, we have two sub-cases, (2a) and (2b), and the choice between Cases (2a) and (2b) is made by the Falsifier. If the Falsifier chooses (2a), then we conclude $M, v \not\models \psi$ for any $v \in R[w]$. If the Falsifier chooses (2b), then by the induction hypothesis, $M, t \models \psi$. The choice between the cases 1 and 2 is made by the Verifier whereas the choice between the cases (2a) and (2b) is made by the Falsifier. Putting all these together with the induction hypothesis, we conclude that $M, w \not\models \psi$ or, $\forall v.wRv$ implies ($v \not\models \psi$ and ($\exists t.vRt$ such that $t \models \psi$)). Finally, by the semantics of the $\triangle$ modality, we have $M, w \models \triangle\psi$. This concludes to strategies-to-truth direction.

The case for $\varphi = \nabla\psi$ is almost identical. Moreover, the cases for $\varphi = \blacktriangle\psi$ and $\varphi = \blacktriangledown\psi$ are similar.

This completes the proof.

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