Chapter 5
Proofs and Refutations, Non-classically and Game Theoretically

Can Başkent

Abstract Lakatos's seminal work *Proofs and Refutations* depends heavily on counter-examples and refutations. In this work, I argue that the said dependancy goes further than anticipated, rendering *Proofs and Refutations* a working example of paraconsistent reasoning in mathematical methodology. I also maintain that *Proofs and Refutations* is an example of paraconsistent reasoning with strategies, making it an example of game theoretical and strategic reasoning in mathematical methodology.

Keywords Proofs and refutations · Lakatosian methodology · Paraconsistency · Game theory

5.1 Introduction

In Lakatosian epistemology contradictions promote knowledge growth. Lakatos, particularly in *Proofs and Refutations*, suggests various methods to eliminate inconsistencies: *monsters* help us to revise the given theory by following a dialectic heuristic, *proofs that do not prove* allow us to revise mathematical theorems or conjectures (Lakatos, 2015, 1979).

However, Lakatos seems to have missed one thing: Whilst the method of proofs and refutations carries out the aforementioned procedures to *maintain* a consistent theory of scientific inquiry, it still needs to *work with inconsistencies* at the object level. Proof attempts, inquiries and lemmas may turn out inconsistent, despite the fact that the meta-theory is committed to maintain the consistency of the theory. Yet, not "everything goes" once such inconsistencies or contradictions are identified. The method of proofs and refutations, like many revisionist methodologies, has some particular rational procedures to follow to revise the theory at hand.

C. Başkent (✉)
Department of Computer Science, Middlesex University, London, UK
e-mail: c.baskent@mdx.ac.uk, https://www.canbaskent.net/logic

R. Frigg et al. (eds.), *Proofs and Research Programmes: Lakatos at 100*, Synthese Library 498, https://doi.org/10.1007/978-3-031-88213-5_5

This is what I will address in this paper, and argue that, intrinsically, the Lakatosian method of proofs and refutations, exemplified in his masterpiece *Proofs and Refutations*, is an inconsistency-friendly system.[1] Moreover, I will show that the inconsistency-friendliness of *Proofs and Refutations can* and *should* be approached game theoretically as it resorts to strategic and rational reasoning with inconsistencies. In short, I argue that the Lakatosian method is *both* paraconsistent and game-theoretical.

There are a number of reasons to think so. First, the Lakatosian method is dialectical (Larvor, 1998).[2] Furthermore, dialectic is perhaps one of the major points of intersection between paraconsistency (or dialetheism) and the Lakatosian thought. As discussed by Ficara, Hegelian dialectic has some strong dialetheic tones (Ficara, 2013).[3] Similarly, it is widely argued that the Lakatosian method converges to Hegelian dialectic (Kvasz, 2002; Musgrave & Pidgen, 2021). Musgrave and Pidgen maintain that

> [T]hese water-tight deductions from well-defined premises are the (perhaps temporary) *end-points* of an evolutionary, and indeed a *dialectical*, process in which the constituent concepts are initially ill-defined, open-ended or ambiguous but become sharper and more precise in the context of a protracted debate.
> (Musgrave & Pidgen, 2021, their emphasis)

Therefore, through the common point of a dialectical approach, the Lakatosian method of proofs and refutations has some dialetheic and inconsistency-friendly aspects.

[1] It is important to notice that the Lakatosian revisionism is not the only method that may benefit from an inconsistency-friendly approach. The Lakatosian method is similar to Hintikka's interrogative models of inquiry in some ways, and it may be helpful to draw some analogies between the two (Başkent, 2015a, 2015b; Başkent, 2017).

[2] Larvor indicates that "(...) Lakatos expressed a desire to become the founder of a dialectical school in the philosophy of mathematics" in a letter written to Larvor (1998, p. 9).

[3] The debate on the Law of Contradiction and Hegel is an illuminating one, as Ficara notes:

> "Classically, interpretations of Hegel's dialectics either take Hegel's claims against the law of non-contradiction (LNC) as a serious logical argument, and therefore do not take Hegel's philosophy seriously, or consider Hegel's philosophy as a serious enterprise, and therefore deny that his critique of LNC should be taken seriously.
> According to a widespread view, whose most authoritative exponent is probably Karl Popper, Hegel's dialectic is unscientific because it implies a refusal of LNC. Popper writes:
>
> > [Hegel's idea of the fertility of contradictions] amounts to an attack upon the 'law of contradiction' [...] of traditional logic, a law which asserts that two contradictory statements can never be true together, or that a statement consisting of the conjunction of two contradictory statements must always be rejected as false on purely logical grounds
>
> For this reason: 'If we are prepared [like Hegel] to put up with contradictions, criticism, and with it all intellectual progress, must come to an end'. And on a similar line, Charles Sanders Peirce observes: 'As far as I know, Hegelians profess to be self-contradictory'."

(*See Ficara's paper for the full reference information for the quotes.*)

Second, the Lakatosian method *needs* and therefore justifies the existence of inconsistencies. What follows from an inconsistency is certainly not *everything* unlike in classical logic. Particularly in *Proofs and Refutations* (PR, for short), Lakatos offers a wide variety of tools to guide what *should* follow from an inconsistency. Such methods include *monster-barring*, *exception-barring*, *method of surrender* and *lemma-incorporation* to name a few (Başkent & Bağçe, 2009). What is common in all these is that they rely and depend on the existence of inconsistencies. For Lakatos, the existence of inconsistencies in a rational theory is not surprising, his method of proofs and refutations relies on their existence. In common with many other revisionists, Lakatos offers certain methods to *fix* the theory. Yet, whilst doing so, the theory works with the very inconsistencies.[4]

Third, in PR, Lakatos identifies certain methods to deal with inconsistencies that show up throughout mathematical practice. These methods are perhaps first there to describe and maintain a classical and consistent theory. Yet, they also serve an important strategic goal, which makes them game-theoretical. This is certainly evident in PR which was written as a sometimes competitive, sometimes cooperative game. Students (that are players) have various strategies that Lakatos identifies, their strategies depend on what they know or learn from other players' moves and strategies. The game is also evolutionary, especially after the Teacher's interventions who keeps introducing new *signals* to the game. Classically, the game identifies rationality with consistency and proceeds as such. However, in due time, I will argue that this is a mistake as it is an unnecessary restriction on the Lakatosian method.

The current paper is organised as follows. First, I briefly discuss the Lakatosian method of proofs and refutations from a paraconsistent point of view. Following, I offer two inter-connected solutions: Lakatosian paraconsistency *and* paraconsistent games for PR. Granted, such solutions are relatively high level, I will conclude with a discussion.

5.2 Lakatos's Method of Proofs and Refutations: Briefly

A brief review of the method of proofs and refutations is a good starting point. Corfield summarises the steps of the Lakatosian method as follows (Corfield, 1997):

[4] An analysis of the nature inconsistencies in the Lakatosian philosophy of mathematics falls outside the scope of the current paper. Many non-classical logicians carefully distinguish inconsistencies and contradictions, and certainly, this approach has some merits (Carnielli & Coniglio, 2016). What is left is to apply it to a particular philosophy of mathematics, such as the Lakatosian method of proofs and refutations.

Moreover, some inconsistencies can be classified as local and global, where the former refutes a local lemma and the latter an overarching theorem. Formal logical approach to such distinctions between inconsistencies, with a direct application to the Lakatosian philosophy of mathematics, remains a challenging future work opportunity.

1. Primitive conjecture.
2. Proof (a rough thought experiment or argument, decomposing the primitive conjecture into subconjectures and lemmas).
3. Global counterexamples.
4. Proof re-examined. The guilty lemma is spotted. The guilty lemma may have previously remained hidden or may have been misidentified.
5. Proofs of the other theorems are examined to see if the newly found lemma occurs in them.
6. Hitherto accepted consequences of the original and now refuted conjecture are checked.
7. Counterexamples are turned into new examples, and new fields of inquiry open up.

This algorithm allows us to make many "searches" and, as such, gives us some room to control the parameters. Searching for counterexamples, re-examining proofs and the methods that are developed to turn them into examples are all strategic moves. Moreover, the existence of inconsistencies is embedded in the algorithm: the algorithm reasons with them rather than "exploding" under their existence. Consequently, the above algorithm makes it clear that the reasoning in PR is a game with inconsistencies—a game with "proofs that do not proof", a game with "guilty lemmas".[5]

If PR enjoys paraconsistent and game theoretical reasoning, then we can use this idea to further our discussion of rationality. What distinguishes game theoretical agents from, say, automata or probabilistic and randomised guesses, is that first and foremost they are rational. As such, they aim at increasing their own pay-offs and maximising their utilities, and consequently winning the game. In order to reach that goal, players need to be allowed to enjoy inconsistent reasoning in their rational strategies. This practice is more common than it seems (Ariely, 2008, 2010; Kahneman, 2011). *Homo economicus* is assumed to be rational yet makes emotional decisions based on inconsistencies in a *systematic* and *predictable* way. Seen as a game, the *game* of mathematical discovery and practice has the potential to share this approach. One can, therefore, imagine a dialogue similar to PR where the players adopt not a revisionist, but a game theoretical approach to mathematical discovery. Instead of discussing how to revise a proof that does not prove, they take turns and make moves to develop a proof that works—even paraconsistently. They may discuss their preferences, pay-offs and strategies to reach an equilibrium in

[5] The way the Lakatosian method resolves inconsistencies shows some notable similarities to Hintikkan method of interrogative inquiry (IMI, for short). IMI is a well-known example of an epistemic method that may result in knowledge increase (Başkent, 2016b). It excludes inconsistencies by *bracketing* them—that is some pieces of information are excluded from the epistemic reasoning as they may lead to contradictions. The decision to choose what to bracket needs to be strategic and rational. This renders IMI also as a game with inconsistencies—a game with "bracketing".

discovering mathematical knowledge. They may talk about “cheaper” proofs which use less resources.[6]

If PR is paraconsistent,[7] then we have a problem. And, this problem requires a solution. I will offer two. First, the theory must allow us to work with inconsistencies with whatever meta-theoretical commitments one might have. Second, the way we progress or resolve the inconsistencies must be strategic in the sense that the players (or rational agents) must be able to *compute* their responses under uncertainty or imperfect information. Furthermore, players must know when the game reaches a solution, or an equilibrium—a state of balance where all players involved are satisfied enough not to make a further move.

In conclusion, the problem of having inconsistencies in PR requires a solution that is *friendly* to contradictions and strategic reasoning as it is the way that the dialogue (the method of proofs and refutations) is presented in PR. In what follows, I will explain how.

5.3 Solution: Paraconsistency for the Lakatosian Method

I have argued earlier that Lakatos’s theory of PR is dialetheic. This means that the theory admits true contradictions. Moreover, I claim that the formal system in which PR seems to be operating in is paraconsistent.

First, as we explained earlier, the Lakatosian method is dialectical (Musgrave & Pidgen, 2021). This idea can be supported by various historical and even political arguments, as many maintained (Corfield, 1997; Koetsier, 1991; Kvasz, 2002; Larvor, 1998). For example, Corfield is critical of Lakatos’s dialectic,[8] Koetsier argues that PR is a rational reconstruction of history,[9] Kvasz criticises Lakatos for

[6] Considering the computational cost of a proof is a well-known approach in computer science. Such costs may include the time it takes to develop a proof, the memory space or the processor power that it requires to compute a proof. Therefore, an agent may have a strict preference towards “cheaper” proofs.

[7] It needs to be noted that arguing that PR is paraconsistent does not suggest that Lakatos himself is a paraconsistent logician nor a dialetheic thinker. The current paper focuses only on PR and leaves it for future work how the Lakatosian philosophy may benefit from dialetheic and paraconsistent approaches.

[8] “[. . .] an important part of the dialectical process is being missed in that good intuitive ideas, which are often the material for the most fruitful variety of rigorous exploration, are being drowned in a sea of conjectures from which they may only be extracted by great effort.” (Corfield, 1997).

[9] “(...) there is no doubt that *Proofs and Refutations* contains a highly counterfactual rational reconstruction” (Koetsier, 1991).

his "confusion" of logic and dialectic,[10] and Larvor's approach is more historical.[11] Such inconsistencies exist in PR, arguably for the purpose of reconstructing the history of Euler's theorem on polyhedra in order to construct a dialectic theory.

Furthermore, Larvor underlines the difference between a logical contradiction and a Hegelian one within the context of the Lakatosian methodology of mathematics.

> For something to contain a contradiction does not mean, for Hegel, that it entails both A and $not-A$ for some proposition A. A Hegelian 'contradiction' is better understood as an internal tension. What it means is that the elements of the object grate against each other in some sense appropriate to the kind of object in question. Now, a Lakatosian research programme is a dynamic unit. Its constituent parts interact and modify each other (in particular, the hard core and heuristic combine to act on the 'protective belt'). It may not even be possible to characterise one part of a research programme in isolation from the others. In the jargon, the parts are essentially related. A research programme is in this sense an organic whole. A programme 'contains a contradiction' (in the Hegelian sense) when it becomes unable to protect its hard core without violating the spirit of its positive heuristic. (Larvor, 1998, p. 70).

However confused or historically motivated he might be, Lakatos's methodology has some strong dialectic tones. Does it, however, suffice to make it *dialetheic*?

Priest argues strongly that dialectic theories are dialetheic (Priest, 1989). He goes further and claims that the use of dialectic by first Hegel and later Marx *requires* dialetheism [ibid, my emphasis]. Recently, Priest also suggested a dialetheism based formalisation of dialectic (Priest, 2023).[12] Ficara, in particular, argues that Hegelian dialectic can be *re*-interpreted as dialetheism. The Hegelian debate on the details of such arguments falls outside the scope of this paper. However, from a dialetheist position, Hegelian dialectic *is* dialetheist. And the logic of dialetheism is paraconsistency—a logic where inconsistencies do not entail everything[13] (Priest, 2002).

[10] "We would like to show that these weaknesses were caused by his confusion of dialectic with logic. In this way, Lakatos developed an appealing and interesting theory, which, at least at first glance, has the advantages of both—the liveliness of dialectic and the soundness of logic. Unfortunately, this attempt to combine dialectic with logic also has one disadvantage. The focus on logic restricts severely the scope of the changes, to which this method can be applied. That is why Lakatos was forced to neglect in his rational reconstruction many episodes in the history of mathematics, which simply do not fit into his scheme. But on the other hand, dialectic gives his theory the illusion of universality, for which reason, perhaps, he seemed to be unaware of his omissions." (Kvasz, 2002).

[11] "The use of historical narrative as philosophical argument is part of Lakatos' Hegelian inheritance. For Hegel, history is a bit like a huge Platonic dialogue. Just as a dialogue starts with simple ideas and progresses dialectically towards a sophisticated conception of whatever happens to be under discussion, so the history of humanity begins with simple forms of consciousness and develops towards a perfect final state." (Larvor, 1998, p. 65).

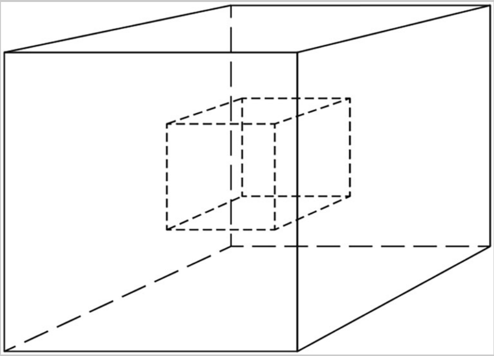
[12] This paper appeared in a special issue of the journal "History and Philosophy of Logic" which was dedicated to various formalisations of dialectic.

[13] Direct applications of paraconsistency include ontology, mereology and belief revision (Priest, 2001).

Therefore, PR is dialectic. As a dialectic theory, it is dialetheic. As such, it must admit a logic of dialetheism, which is the paraconsistent logic. This is an interesting direction to take. Because, as Larvor argued above, it is not possible to reach the conclusion that "PR is paraconsistent" by means of Hegel in the way that Lakatos understood him, as Hegel's interpretation of contradictions is not entirely logical. Instead, we reach this conclusion by means of dialectic and dialetheism.

Let us now consider an example from PR to illustrate Lakatos's interpretation of reasoning with inconsistencies.

ALPHA: I have a counterexample which (...) will be a counterexample to the main conjecture, i.e. this will be a global counterexample as well. (...)
Imagine a solid bounded by a pair of nested cubes—a pair of cubes, one of which is inside, but does not touch the other. This hollow cube falsifies your first lemma, because on removing a face from the inner cube, the polyhedron will not be stretchable on to a plane. Nor will it help to remove a face from the outer cube instead. Besides, for each cube $V - E + F = 2$, so that for the hollow cube $V - E + F = 4$.
(...)
GAMMA: (...) Hands up! You have to surrender. Scrap the false conjecture, forget about it and try a radically new approach.
(...)
DELTA: But why accept the counterexample? We proved our conjecture—now it is a theorem. I admit that it clashes with this so-called 'counterexample'. One of them has to give way. But why should the theorem give way, when it has been proved? It is the 'criticism' that should retreat. It is fake criticism. This pair of nested cubes is not a polyhedron at all. It is a *monster*, a pathological case, not a counterexample.
(Lakatos, 2015, p. 14-5, Lakatos's emphasis)



Following this dialogue, PR continues with a discussion of *monster-barring*. The method of monster-barring suggests that the object, which has been put forward as a counter-example, is actually not a counter-example but a *pathological case* (Lakatos, 2015, p. 15).

When the nested-cube contradicts the Conjecture, not everything follows proof-theoretically. Certain strategies are employed in order to reach a temporary state of consistency or even an equilibrium. In the example above, this strategy is monster-barring. Later on, PR suggests another strategy, called *exception-barring*. The meta-theory of PR, therefore, strictly suggests that not-everything goes. Only *some* strategies to resolve the puzzle can be employed.

The contradiction in the example above is worth discussing.

The conjecture suggests that $\forall x.\chi(x) = 2$ where x varies over a domain of geometric objects and $\chi(x)$ represents the Euler characteristics of object x. The existence of the nested-cube, denoted by n, on the other hand, shows that $\chi(n) = 4$. And, as $2 \neq 4$ and n lies within the domain of geometric objects, the following two sentences are inconsistent.

$$\forall x.\chi(x) = 2$$

$$\chi(n) \neq 2$$

It is important to note that Lakatos's end goal is not to leave the state of affairs there, but resolve the contradiction. This, however, still does not refute our claim that the system remains paraconsistent. What follows or does not follow from a contradiction defines the Lakatosian methodology—if everything did follow, there would be no Lakatosian methodology. All would be trivial. But, they are not.

Let me clarify this argument further.[14] Paraconsistent reasoning suggests a rational way of reasoning under inconsistencies. Priest, for example, discusses Bohr's theory of atom, Dirac's δ-function and Newton/Leibniz's infinitesimal calculus as examples of inconsistent scientific theories (Priest, 2007). Yet, one needs to "reason in a non-trivial way from inconsistent information" [ibid]. And this way of thinking can also be the rational way of reasoning. Priest argues that "it seems reasonable to hold that if one theory is sufficiently better than all of its competitors on sufficiently many of the criteria, then, rationally, one should believe this rather than the others." [ibid].

Bueno and da Costa argue along the same lines, and defend the "view that if scientific theories are taken to be quasi-true, and if the underlying logic is paraconsistent, it's perfectly rational for scientists and mathematicians to entertain inconsistent theories without triviality. As a result, as opposed to what is demanded by traditional approaches to rationality, it's not irrational to entertain inconsistent theories" (Bueno & da Costa, 2007).

---

[14] I am thankful to the anonymous referee for asking for further clarifications for this argument.

Therefore, I maintain that it is possible that one can be rational whilst reasoning with inconsistencies. Furthermore, the Lakatosian method of proofs and refutations is an example of this. Opponents of this thesis may maintain that this can also be handled within the realm of classical logic. This, however, is far from true. The reason is that the Lakatosian method relies on inconsistencies. If there were no inconsistencies, there would be no Lakatosian method. If there were no inconsistencies, there would be no "guilty lemmas", no "proofs that do not prove". This defines the Lakatosian methodology. If all followed from a contradiction, it would be impossible to identify what characterises Lakatosian methodology. This is the reason why there would be no Lakatosian methodology in a classical logic.

Finally, one can argue that, syntactically, the Lakatosian contradictions in PR are conditional contractions, therefore can easily be handled within the power of classical logic and belief revision methodologies.[15] This argument, however, is not necessarily true. If a contradiction appears to follow from $\varphi_1 \wedge \cdots \wedge \varphi_n$, and if removing φ_1 from the conjunction eliminates the contradiction, then one can argue that this is good enough to maintain a consistent theory. There are, however, some problems with this approach. First, the theory loses information as φ_1 is now *not* part of the conjunction. There may be some informational cost attached to removing a conjunct. Second, it is not always possible to identify the "trouble-maker conjunct" as it may not be theoretically possible.[16] Three, the elimination based methods to maintain logical consistency argue not only against paraconsistency but also against non-classical conjunctions. One can desire to maintain consistency, but this does not immediately entail that one needs to stick with classical conjunction *and* classical entailment. They are often separate issues. Moreover, one can also maintain a classical theory by introducing different conjunctions (including multiplicative conjunctions, for example) or conditionals (including the magic-wand operator in separation logics, for example). In conclusion, there are various logical *tricks* that one can use to maintain a classical theory. Yet, they do not refute our central arguments as to why PR needs a paraconsistent theory. Because the reason is not the lack of logical and mathematical methods to eliminate inconsistencies. The reason is that PR needs inconsistencies to operate.

In PR, Lakatos offers various methods to work with inconsistencies. And these methods characterise PR. Let us now consider another example which reiterates some of the points I have made earlier.

GAMMA: I have just discovered that my *Counterexample 5*, the cylinder, refutes not only the naive conjecture but also the theorem. Although it satisfies both lemmas, it is not Eulerian.

(continued)

[15] I am thankful to the anonymous referee for pointing this out.

[16] In Priest (2007), Priest list some examples where it is not simply possible to follow the elimination strategy in some scientific theories.

ALPHA: Dear Gamma, do not become a crank. The cylinder was a joke, not a counterexample. No serious mathematician will take the cylinder for a polyhedron.
GAMMA: Why didn't you protest against my *Counterexample 3*, the urchin? Was that less 'crankish' than my cylinder? *Then* of course you were *criticising* the naive conjecture and welcomed refutations. Now you are *defending* the theorem and abhor refutations! *Then*, when a counterexample emerged, your question was: *what is wrong with the conjecture*? *Now* your question is: *what is wrong with the counterexample*?
DELTA: Alpha, you have turned into a monster-barrer! Aren't you embarrassed?
(Lakatos, 2015, p. 45, Lakatos's emphasis)

The above (counter-) example, *cylinder*, once again, is used to establish a methodology to ward off contradictions. Using the terminology above, now we have the following, where c denotes cylinder.

$$\forall x.\chi(x) = 2$$
$$\chi(c) \neq 2$$

The method of monster-barring argues that c does not lie within the scope of the universal quantifier $\forall x$, hence the theorem would not apply to c. Once again, we have a rational and logical strategy to follow when contradictions emerge. Not everything goes.

As Lakatos puts it in PR, "*[t]his revolution in mathematical criticism changed the concept of mathematical truth, changed the standards of mathematical proof, changed the patterns of mathematical growth!*" (Lakatos, 2015, p. 110–111, Lakatos's emphasis). This change, however, may not work the way Lakatos imagined.

Non-classical, in particular paraconsistent, reading of the Lakatosian method is arguably another point of support for Lakatos's view regarding mathematics as quasi-empirical. Lakatos asked a similar question when he was discussing falsification in mathematics: "[W]hat is the nature of potential falsifiers of mathematical theories?" (Lakatos, 1979, p. 35). His immediate answer is amusing but indeed accurate: "The very question would have been an insult in the years of intellectual honeymoon of Russell or Hilbert. After all, *Principia* or the *Grundlagen der Mathematik* were meant to put an end—once and for all—to counterexamples and refutations in mathematics." (Lakatos, 1979, p. 36). Granted, Lakatos's understanding of formal theories and their quasi-empirical counterparts is not paraconsistent. Nevertheless, his reasoning, the way the method of proofs and refutations work, is. Counterexamples and refutations are essential parts of his methodology and the theory of PR relies on them. This makes it paraconsistent—reasoning with inconsistencies, refutations, contradictions and counter-examples in a non-trivial way.

The opponents of this idea may suggest that there are subtle differences between inconsistencies and counter-examples. The former is a death sentence for a theory, whereas the latter allows us to improve the theory, promoting knowledge growth. From a classical point of view, there is some truth in this line of thought. After all, working with inconsistencies in a theory provides us with feedback to *fix* the theory. Yet again, the way the theory can be *fixed* or improved depends on the meta-theory of the methodology one needs to endorse amongst others. One way or another, revisionist theories still work with inconsistencies—until the next counter-examples emerge. Furthermore, the existence of the plethora of revisionist theories for mathematical practice is yet another argument supporting paraconsistency. Once inconsistencies emerge, there may be more than one way of fixing the theory, if that is the interim goal. Not everything goes as revisionists might argue, but only a selected few methodologies.

A similar approach has been taken by paraconsistent mathematicians and philosophers (Mortensen, 1994, 2010; Weber, 2021). Particularly, Weber's recent work develops a basic theory of sets that is "axiomatically in a paraconsistent logic", and extends the discussion to a few other foundational issues in mathematics, including algebra and topology. Therefore, once met with the paradoxes of set theory, one can choose to follow the Euclidean methodology or the Lakatosian one or the paraconsistent one, amongst others.[17] One thing in common with all such methods is, however, that all work with inconsistencies. Not everything follows from a contradiction or a paradox.

In PR, Lakatos offers many methods and techniques to understand contradictions. The way Lakatos attempts to *resolve* contradictions is, however, game theoretical.

5.4 Solution: Paraconsistent Games for the Lakatosian Method

With a slight abuse of terminology, it is possible to define a paraconsistent game: a game where, under contradictions, players are able to make rational moves to increase their pay-offs and form rational strategies. There are various ways a game can be *paraconsistent*. It can be about what players know—epistemic paraconsistent games. It can also be about inconsistent strategies and moves—behavioural paraconsistent games. In this work, when I say paraconsistent games, I will use it to mean either of the aforementioned games.

[17] This is a good point for logical pluralism, as discussed by Beall and Restall (2006). Nevertheless, in order not to diverge from my focus, I leave it for a future work to examine the logical pluralism of the Lakatosian thought.

In game theoretical paraconsistency, contradictory strategies (strategies that can lead to both a win and a loss), irrational players (players who do not seek to maximise their utility) and non-classical probabilities are allowed (Bueno-Soler & Carnielli, 2016; Mares, 1997). The aforementioned elements of inconsistencies are also familiar from game semantics which can be a good example of a paraconsistent game. Hintikkan game semantics suggests an interpretation of logical formulas by means of game theoretical choices and operations. As such, loosely put, it *gamifies* logic (Hintikka & Sandu, 1997; Mann et al., 2011).[18] Semantic games, depending on the logics in question, can be paraconsistent: we can have competitive games where both players may admit winning strategies. We can even have a cooperative, multi-player and coalition-based games for various non-classical logics (Başkent, 2016a, 2020; Başkent & Henrique Carrasqueira, 2020). Arguably, paraconsistent games are not a foreign concept in philosophical logic, and in what follows I will show that such elements also exist in PR by focusing on individual tools and techniques employed in PR.

First, *proofs that do not proof* are the proof theoretical equivalent of the game theoretical strategies that *knowingly* produce a loss. Even if they may not bring a win, players may still learn from the plays following such strategies, thus there may be an epistemic gain despite the loss. There might, however, be a cost to this: computational power and resources may be consumed whilst producing a proof that does not prove. Moreover, such proofs may produce a signal: an error in the proof might signal another error in a lemma, for instance. This also makes them a subject of evolutionary game theory.[19] From a paraconsistent perspective, proofs that do not prove is another argument to support paraconsistency. These are the proofs from which some propositions do not follow—even if there is an inconsistency. This is paraconsistency.

On the other hand, considering the constructivist Kolmogorov connection between "truth–proofs–computation–strategies", there is much more to say about proofs that do not prove. Such objects, if they exist, are the strategies that do not bring wins and the programs that do not compute what they are set out to compute. Inconsistent (or dialetheic) truth suggests "proofs that do not prove" exist, which in turn suggest that "programs that do not compute" exist, which finally suggest "winning strategies that do not bring a win" exist as well. The theories of the latter two are not well developed yet. Nevertheless, proofs that do not prove show the existence of such objects by means of the aforementioned Kolmogorov connection.

Second, *re-examining proofs* is strategy pruning. Similar to well-known game theoretical situations, such as the iterated prisoners' dilemma, in strategy pruning,

[18] It is important that the research programme of gamifying logic did not start with Hintikka, and can be seen in some works of Hodges (2013).

[19] Games where players knowingly follow a strategy that generate a loss are a common subject of inquiry in evolutionary game theory. Altruism, both in animal and non-animal groups is a well-known example.

the player re-runs the strategy and uses it to improve the same initial strategy by simply removing those alternatives that players know would not work.

Another major example is the “iterated elimination of strictly dominated strategies”—a common solution method in game theory. In this method, the strategies that are eliminated can be the pruned version of the very strategy. In other words, when a given strategy σ is pruned to σ' and played along in the next run of the game, the strictly dominated strategy amongst σ and σ' is eliminated. The strategies can be refined, programs can be made efficient. This, however, contradicts the basic tenets of strategies (Hodges, 2013; van Benthem & Klein, 2022). A strategy is supposed to be pre-defined. Therefore, revising a strategy must already be part of the strategy. This creates self-reference as I argue next.

Third, *revising proofs* is strategy revision, which contradicts the very definition of a strategy. A strategy is defined as “a set of rules that describe exactly how [a] player should choose, depending on how the [other] players have chosen at earlier moves” (Hodges, 2013). That means that strategies are pre-set and pre-defined. They are omniscient and should cover all possible cases and scenarios, including their own revision. Hence, a strategy cannot be revised, it should contain its own revision. This is self-reference (Heifetz, 1996). Harsanyi noted this much earlier in 1967:

> It seems to me that the basic reason why the theory of games with incomplete information has made so little progress so far lies in the fact that these games give rise, or at least appear to give rise, to an infinite regress in reciprocal expectations on the part of the players. In such a game player 1’s strategy choice will depend on what he expects (or believes) to be player 2’s payoff function U_2, as the latter will be an important determinant of player 2’s behavior in the game. But his strategy choice will also depend on what he expects to be player 2’s first-order expectation about his own payoff function U_1. Indeed player 1’s strategy choice will also depend on what he expects to be player 2’s second- order expectation – that is, on what player 1 thinks that player 2 thinks that player 1 thinks about player 2’s payoff function U_2... and so on *ad infinitum* .
> (Harsanyi, 1967)

Heifetz argued similarly on the same issue:

> Nevertheless, one may continue to argue that a state of the world should indeed be a circular, self-referential object: A state represents a situation of human uncertainty, in which a player considers what other players may think in other situations, and in particular about what they may think there about the current situation. According to such a view, one would seek a formulation where states of the world are indeed self-referring mathematical entities.
> (Heifetz, 1996)

There is a wide variety of work on self-referential paradoxes in games, including their connections to *non*-self-referential variations (Abramsky & Zvesper, 2015; Brandenburger & Keisler, 2006; Pacuit, 2007; Başkent, 2018). The mathematical details of such ideas, however, fall outside the scope of the current work.

Fourth, *turning contradictions into new examples* and *lemma-incorporation* are methods familiar from belief revision and dynamic logics (Mares, 2002; Priest, 2001). However, for PR it is an embedded and essential part of the heuristic method. It is a meta-model theoretical strategy that changes and updates the model where inconsistencies become consistencies. One can have different metaphysical goals of obtaining a consistent strategy at the end. Nevertheless, as before, the very act of reasoning game theoretically and strategically under the inconsistencies is paraconsistent.

There is yet another angle. The epistemics of players and strategies can be analysed further using epistemic game theory. For instance, what players know about each other and their preferences and how they revise their strategies after a new piece of information is introduced, fall within the scope of epistemic game theory. PR is no exception to this approach.

Epistemic game theoretical elements in PR are plentiful. In the following passage from PR, some epistemic elements and the preferences based on them are revealed step by step.

> GAMMA: (...) *A polyhedron is a solid whose surface consists of polygonal faces*. And my counterexample is a solid bounded by polygonal faces.
> TEACHER: Let us call this definition *Def. 1*.
> DELTA: Your definition is incorrect. A polyhedron must be a surface: it has faces, edges, vertices, it can be deformed, stretched out on a blackboard, and has nothing to do with the concept of 'solid'. *A polyhedron is a surface consisting of a system of polygons*.
> TEACHER: Call this *Def. 2*.
> DELTA: So really you showed us two polyhedra—two surfaces, one completely inside the other. A woman with a child in her womb is not a counterexample to the thesis that human beings have one head.
> (...)
> DELTA: (...) By polyhedron I meant *a system of polygons arranged in such a way that (1) exactly two polygons meet at every edge and (2) it is possible to get from the inside of any polygon to the inside of any other polygon by a route which never crosses any edge at a vertex*. (...)
> TEACHER: *Def. 3*.
> ALPHA: (...) Why don't you just define a polyhedron as a system of polygons for which the equation $V - E + F = 2$ holds? This Perfect Definition ...
> KAPPA: *Def. P*.
> (Lakatos, 2015, p. 15-7, Lakatos's emphasis)

Above, players reveal what they know and understand about polyhedron and what they think and believe about each other's knowledge about polyhedron. It is interactive and it is strategic.[20]

PR presents some more instances of epistemic games where players learn from each other and develop their ideas further. In such instances what they know

[20] Lakatos's subsection titles in PR make the classification about the role of strategies very easy to follow.

about their opponents and what they know about what their opponents know about themselves change, improve and evolve. The way they execute their strategies and make moves directly depends on how their knowledge about the aforementioned situations is formed.

Let us now see some examples from PR in order to illustrate the game theoretical elements at work.

First, the method of exception-barring.

> BETA: (...) It now seems to me that no conjecture is generally valid, but only valid in a certain restricted domain that excludes the *exceptions*. I am against dubbing these exceptions 'monsters' or 'pathological cases'. That would amount to the methodological decision not to consider these as interesting *examples* in their own right, worthy of a separate investigation. But I am also against the term '*counterexample*'; it rightly admits them as examples on a par with the supporting examples, but somehow paints them in war-colours, so that, (...), one panics when facing them, and is tempted to abandon beautiful and ingenious proofs altogether. No: they are just *exceptions*.
> (Lakatos, 2015, p. 26, Lakatos's emphasis)

The above quote discusses which moves are or can be admissible for players at certain positions in the game. Following PR, if some "counterexamples" are to be excluded, this simply means that there is no available move in that position in the game that allows the players to admit those objects as counterexamples. They are, then, "exceptions", and excluded.

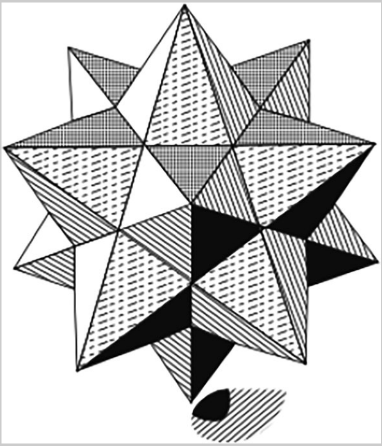
Moreover, if some moves are not available, we can then discuss *admissible strategies*. A strategy s is admissible if and only if there is a strictly positive probability measure on the strategy profiles for the other players, under which s is optimal (Brandenburger et al., 2008). Admissibility is important as it is "a prima facie reasonable criterion: It captures the idea that a player takes all strategies for the other players into consideration; none is entirely ruled out" (Brandenburger et al., 2008). This concept explains exception-barring well. In order to identify a geometric object as an exception, it must be compared and contrasted against the counterexamples, without immediately being ruled out. Moreover, for some pupils in PR, exception-barring as a strategy must be optimal.

Second, the method of monster-adjustment.

> RHO: I agree that we should reject Delta's monster-barring as a general methodological approach, for it doesn't really take 'monsters' seriously. Beta doesn't take his 'exceptions' seriously either, for he merely lists them and then

(continued)

retreats into a safe domain. Thus both these methods are interested only in a limited, privileged field. *My* method does not practise discrimination. I can show that 'on closer examination the exceptions turn out to be only apparent and the Euler theorem retains its validity even for the alleged exceptions'.
(...)
ALPHA: How can my counterexample 3, the 'urchin', be an ordinary Eulerian polyhedron? It has 12 star-pentagonal faces...
RHO: I don't see any 'star-pentagons'. Don't you see that in actual fact this polyhedron has ordinary triangular faces? There are 60 of them. It also has 90 edges and 32 vertices. Its 'Euler characteristic' is 2. The 12 'star-pentagons', their 30 'edges' and 12 'vertices', yielding the 'characteristic'—6, are only your fancy. Monsters don't exist, only monstrous interpretations. One has to purge one's mind from perverted illusions, one has to learn how to see and how to define correctly what one sees. My method is therapeutic: where you—erroneously—'see' a counterexample, I teach you how to recognise—correctly—an example. I adjust your monstrous vision...



(Lakatos, 2015, p. 33, Lakatos's emphasis)

Monster-adjustment then is a case of evolutionary game theory. Evolutionary games are about change. Species can evolve, norms may change, definitions can be revised. Evolutionarily, a predator can evolve to be a prey in a different eco-system. And, as such, monsters can be adjusted.

Third, the method of monster-barring.

GAMMA: *Then* of course you were *criticising* the naive conjecture and welcomed refutations. *Now* you are *defending* the theorem and abhor refutations! Then, when a counterexample emerged, your question was: *what is wrong with the conjecture*? *Now* your question is: *what is wrong with the counterexample*?
(Lakatos, 2015, p. 45, Lakatos's emphasis)

Winning a game, that is proving a proposition φ, does not always mean disproving $\neg\varphi$. Logic must allow this and must have a non-classical negation. Similarly, the *game* of proving a proposition does not have to be zero-sum. Hintikka's game theoretical semantics is a good example of this once used for non-classical logics, as I argued earlier (Başkent, 2016a). For some non-classical logics, for example, winning a game (showing the truth of a proposition φ) does not entail that the opponent loses (thus $\neg\varphi$ does not have to fail).

Similarly, in PR, defending the conjecture and refuting the counterexamples do not necessarily suggest the same strategy. This is another way of seeing non-classical logical elements at work in PR—both semantically and game theoretically.

Fourth, the method of proofs and refutations.

LAMBDA: All this shows that one cannot put proof and refutations into separate compartments. This is why I would propose to rechristen our '*method of lemma-incorporation*' the '*method of proof and refutations*'. Let me state its main aspects in three heuristic rules:
Rule 1. If you have a conjecture, set out to prove it and to refute it. Inspect the proof carefully to prepare a list of non-trivial lemmas (proof-analysis); find counterexamples both to the conjecture (global counterexamples) and to the suspect lemmas (local counterexamples).
Rule 2. If you have a global counterexample discard your conjecture, add to your proof-analysis a suitable lemma that will be refuted by the counterexample, and replace the discarded conjecture by an improved one that incorporates that lemma as a condition. Do not allow a refutation to be dismissed as a monster. Try to make all 'hidden lemmas' explicit.
Rule 3. If you have a local counterexample, check to see whether it is not also a global counterexample. If it is, you can easily apply Rule 2.
(Lakatos, 2015, p. 53, Lakatos's emphasis)

This is the construction of a (recursive) strategy that directly relies on opponents' moves and strategies. This is in line with Harsanyi and Heifetz's thesis, presented earlier.

Examples can be multiplied. Let us stop here.

Clearly, Lakatos would like these games to terminate: "Unlimited concept-stretching destroys meaning and truth" (Lakatos, 2015, p. 105). This means, game theoretically, that there is a non-zero cost attached to concept-stretching in the method of proofs and refutations. And, it seems that the *reason* for that cost is metaphysical rather than mathematical as "*[r]ationality, after all, depends on inelastic, exact, concepts!*" (Lakatos, 2015, p. 108, Lakatos's emphasis). Lakatos explains it further in a footnote: "Gamma's demand for a crystal-clear definition of 'counterexample' amounts to a demand for crystal-clear, inelastic concepts in the metalanguage as a condition of rational discussion" [ibid].

Nevertheless, a game theoretical perspective remains to be the best way to analyse Lakatos's heuristic recursion. The aforementioned rigidity in the concepts of metalanguage finds some resonance in non-classical and heterodox approaches to game theory by means of behavioural game theory and applied psychology. The way classical and traditional game theory evolved into behavioural game theory resembles how the method of proofs and refutations *may* evolve, breaking the rigid expectations for the ontology of the metalanguage in question to understand rationality. Game theoretical methodologies to dissect PR further can produce a behavioural account of the method of proofs and refutations. Such behavioural and strategic accounts can explain the goal-oriented, opportunist and rational approaches demonstrated by the players (that are the pupils and the Teacher in PR, who often represent historical mathematicians). This would be another line of inquiry to explore the non-classical elements in PR.

5.5 Conclusion

The theoretical richness of PR allows us to approach it from a variety of perspectives (Başkent, 2009, 2012; Başkent, 2015a). A game theoretical analysis of PR, using contemporary logical advancements, builds on this tradition and sheds further light on PR.

What separates the Lakatosian philosophy of mathematics from the others is how it reacts to inconsistencies and how it channels inconsistencies to mathematical discovery. It is important to underline that this procedure is paraconsistent, and can be extended to the debates within some other schools of thought in philosophy of mathematics. One can argue that what we call mathematical discovery is the result of applying a fit-for-purpose paraconsistent logic to inconsistencies at hand. The examples we discussed illustrate how Lakatos's paraconsistent logic works. It is not difficult to apply the same approach to other methods of mathematical discovery in order to unearth the game theoretical elements in them.

Reasoning with strategic inconsistencies and inconsistent strategies allows us to present a new body of evidence for the dialetheic agenda. In addition to epistemological, truth theoretical and logical arguments for it, dialetheism can also be approached from a game theoretical perspective. This paper aims at presenting a case study to establish an argument *for* dialetheism as well as shedding light on the Lakatosian methodology from a non-classical logical perspective.

This work fits within a broader research programme, called "Paraconsistent Games". Behavioural economics, applied psychology and artificial intelligence provide a plethora of cases where rational people (and machines) may behave, reason or think inconsistently in a coherent way. However, it is important to reinforce this line of thought using ideas from the philosophy of practice-based mathematics, particularly the Lakatosian thought. This allows us to see the remits of the aforementioned research programme as well as the not-well-studied aspects of the Lakatosian methodology. Discussing rationality within this context enriches such debates. We leave it for future work.

There is more to be done. An examination of the direct connection between the method of proofs and refutations and dialetheism within the context of practice-based mathematics is an immediate next step. The way that contradictions play a *constructive* role in the generation and discovery of mathematical knowledge is important for dialetheism and formal theories of rationality. Second, on a broader scale, relating PR to logical pluralism by examining the constructivist elements in it remains a big task. Third, it is important to compare and contrast our approach with various other revisionist philosophies of mathematics, including Hintikka's, as we already touched upon. Fourth, as argued earlier very briefly, game theory is the study of strategies and rational behaviour. A "behavioural" approach to PR would allow us to understand the very mathematical methodologies used to construct various proofs of the Euler's Conjecture. Characterising, for instance, Cauchy's approach to the conjecture and its proof using game theoretical elements, and then contrasting it with the strategy followed by, for instance, Lhuilier or Gergonne would shed some light on PR—both historically and mathematically.

Such future work would enrich the debates in both non-classical logic and the philosophy of mathematics, providing a long-overdue connection.

Acknowledgments This paper is a much expanded version of a talk delivered at the Imre Lakatos Centenary Conference at the London School of Economics in November 2022. I am very thankful to the organisers.

I very much appreciate the patient comments of the anonymous referee(s), which helped improve the paper.

A substantially earlier version of this text was presented at a special session on "*How to Solve It?*: Heuristics and Inquiry Based Learning" at the American Mathematical Society's Fall Western Sectional (remote) meeting in October 2020. I am grateful to the organisers for the opportunity.

After his mere 51 years in this world, I still find inspiration and research ideas in Lakatos's work, particularly in *Proofs and Refutations*, which I think is the best gift given to mathematicians from a philosopher.

References

Abramsky, S., & Zvesper, J. (2015). From Lawvere to Brandenburger-Keisler: Interactive forms of diagonalization and self-reference. *Journal of Computer and System Sciences*, *81*(5), 799–812.

Ariely, D. (2008). *Predictably irrational: The hidden forces that shape our decisions*. Harper-Collins.

Ariely, D. (2010). *The upside of irrationality*. Harper.
Başkent, C. (2009). A geometrical - epistemic approach to Lakatosian heuristics. In C. Drossos, P. Peppas, & C. Tsinakis (Eds.), *Proocedings of Seventh Panhellenic Logic Symposium* (pp. 8–14). Patras University Press.
Başkent, C. (2012). A formal approach to Lakatosian heuristics. *Logique et Analyse*, *55*(217), 23–46.
Başkent, C. (2015a). Inquiry, refutations and the inconsistent. *Perspectives on interrogative models of inquiry - Developments in inquiry and questions* (pp. 57–71). Springer.
Başkent, C. (Ed.). (2015b). *Perspectives on interrogative models of inquiry: Developments in inquiry and questions*. Springer.
Başkent, C. (2016a). Game theoretical semantics for some non-classical logics. *Journal of Applied Non-Classical Logics*, *26*(3), 208–239.
Başkent, C. (2016b). Towards paraconsistent inquiry. *Australasian Journal of Logic*, *13*(2), 21–40.
Başkent, C. (2017). A non-classical logical approach to social software. In C. Başkent, L. Moss, & R. Ramanujam (Eds.), *Rohit Parikh on logic, language and society* (pp. 91–109). Springer.
Başkent, C. (2018). A Yabloesque paradox in epistemic game theory. *Synthese*, *195*(1), 441–464.
Başkent, C. (2020). A game theoretical semantics for a logic of nonsense. In J. F. Raskin & D. Bresolin (Eds.), *Proceedings of the 11th International Symposium on Games, Automata, Logics, and Formal Verification (GandALF 2020)* (Vol. 326, pp. 66–81). Electronic Proceedings in Theoretical Computer Science.
Başkent, C., & Bağçe, S. (2009). An examination of counterexamples in *proofs and refutations*. *Philosophia Scientiae*, *13*(2), 3–20.
Başkent, C., & Henrique Carrasqueira, P. (2020). A game theoretical semantics for a logic of formal inconsistency. *Logic Journal of the IGPL*, *28*(5), 936–952.
Beall, J. C., & Restall, G. (2006). *Logical pluralism*. Clarendon Press.
Brandenburger, A., & Keisler, H. J. (2006). An impossibility theorem on beliefs in games. *Studia Logica*, *84*, 211–240.
Brandenburger, A., Friedenberg, A., & Keisler, H. J. (2008). Admissibility in games. *Econometrica*, *76*(2), 307–352.
Bueno, O., & da Costa, N. C. A. (2007). Quasi-truth, paraconsistency, and the foundations of science. *Synthese*, *154*, 383–399.
Bueno-Soler, J., & Carnielli, W. A. (2016). Paraconsistent probabilities: consistency, contradictions and Bayes' theorem. *Entropy*, **18**(9), 325.
Carnielli, W. A., & Coniglio, M. E. (2016). *Paraconsistent logic: Consistency, contradiction and negation*. Springer.
Corfield, D. (1997). Assaying Lakatos's history and philosophy of science. *Studies in History and Philosophy of Science*, *28*(1), 99–121.
Ficara, E. (2013). Dialectic and dialetheism. *History and Philosophy of Logic*, *34*(1), 35–52.
Harsanyi, J. C. (1967). Games with incomplete information played by "Bayesian" players: I. The basic model. *Management Science*, *14*(3), 159–182.
Heifetz, A. (1996). Non-well-founded type spaces. *Games and Economic Behavior*, *16*, 202–217.
Hintikka, J., & Sandu, G. (1997). Game-theoretical semantics. In J. van Benthem & A. ter Meulen (Eds.), *Handbook of logic and language* (pp. 361–410). Elsevier.
Hodges, W. (2013). Logic and games. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy*.
Kahneman, D. (2011). *Thinking, fast and slow*. Farrar, Straus and Giroux.
Koetsier, T. (1991). *Lakatos' philosophy of mathematics: A historical approach*. North-Holland.
Kvasz, L. (2002). Lakatos' methodology between logic and dialectic. In G. Kampis, L. Kvasz, & M. Stölzner (Eds.), *Appraising Lakatos: Mathematics, methodology and the man*. Kluwer.
Lakatos, I. (1979). *Mathematics, science and epistemology*. Cambridge University Press.
Lakatos, I. (2015). *Proofs and refutations*. *Cambridge philosophy classics*. Cambridge University Press.
Larvor, B. (1998). *Lakatos: An introduction*. Routledge.

Mann, A. L., Sandu, G., & Sevenster, M. (2011). *Independence-friendly logic*. Cambridge University Press.
Mares, E. D. (1997). Paraconsistent probability theory and paraconsistent Bayesianism. *Logique et Analyse*, *40*(160), 375–384.
Mares, E. D. (2002). A paraconsistent theory of belief revision. *Erkenntnis*, *56*, 229–246.
Mortensen, C. (1994). *Inconsistent mathematics*. Springer.
Mortensen, C. (2010). *Inconsistent geometry*. College Publications.
Musgrave, A., & Pidgen, C. (2021). Imre Lakatos. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy*.
Pacuit, E. (2007). Understanding the Brandenburger-Keisler belief paradox. *Studia Logica*, *86*(3), 435–454.
Priest, G. (1989). Dialectic and dialetheic. *Science & Society*, *53*(4), 388–415.
Priest, G. (2001). Paraconsistent belief revision. *Theoria*, *67*(3), 214–228.
Priest, G. (2002). Paraconsistent logic. In D. Gabbay & F. Guenthner (Eds.), *Handbook of philosophical logic* (Vol. 6, pp. 287–393). Kluwer.
Priest, G. (2007). Paraconsistency and dialetheism. In D. M. Gabbay & J. Woods (Eds.), *Handbook of history of logic* (1st ed., Vol. 8, pp. 129–204). Elsevier.
Priest, G. (2023). The logical structure of dialectic. *History and Philosophy of Logic*, *44*(2), 200–208.
van Benthem, J., & Klein, D. (2022). Logics for analyzing games. In *The Stanford encyclopedia of philosophy*.
Weber, Z. (2021). *Paradoxes and inconsistent mathematics*. Cambridge University Press.