Logics of spatial isolation

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Abstract

In the vein of recent work that provides non-normal modal interpretations of various topological operators, this paper proposes a modal logic for a spatial isolation operator. Focusing initially on neighbourhood systems, we prove several characterization results, demonstrating the adequacy of the interpretation and highlighting certain semantic insensitivities that result from the relative expressive weakness of the isolation operator. We then transition to the topological setting, proving a topological characterization result.

Keywords: isolated points, topological semantics, neighbourhood systems, Alexandrov spaces, T_0 spaces, T_1 spaces.

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1 Introduction

Topological interpretations provided some of the earliest semantics for modal logics (e.g. [9, 15, 16]). These early interpretations focused on $\Diamond$ as topological closure. Subsequent work demonstrated that $\Diamond$ can also be interpreted as the derivative[1] [3, 5, 9, 12]. More recently, it has been shown that other topological operators—including border and boundary operators—can provide fruitful interpretations of various, usually non-normal, modal operators ([13]).

This paper attempts to continue this more recent line of inquiry by proposing a modal logic for topological isolation, a fundamental concept in point-set topology. When $\mathcal{X} = \langle X, \tau \rangle$ is a topological space and $S \subseteq X$, x is an *isolated point* of S if there is an open neighbourhood U of x such that $U \cap S = \{x\}$. We demonstrate that the $[i]$ operator introduced in [8] (where it is intended to model a notion of factive ignorance) can be spatially interpreted as an isolated points operator.

This work is also related to, and partially motivated by, the project of obtaining a deeper understanding of the semantic insensitivities displayed by modal logics, particularly non-normal

[1] The *derivative* of a set S, $d(S)$, is the set of *limit points* of S.

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modal logics. (In the context of relational frames, $[i]$ is insensitive to modifications of the accessibility relation in which elements of the diagonal relation are either added or removed.)

Hence, this paper can be seen as a contribution to the longstanding research programme concerned with developing modal definitions for topological notions as well as topological explanations for modal phenomena (such as modal insensitivity). While the main goal is to present a study of the model theory of the modal logic of topological isolation, this stands as a first step towards studying similar topological concepts, including topological separation axioms.

The paper is organized as follows. In §2, the basic syntax, axiomatic system and relational semantics are introduced briefly. Due to the relative lack of algebraic structure of the isolated points operator, instead of immediately focusing on topologies, we begin, in §3, by considering neighbourhood systems.[2] As neighbourhood systems are a generalization of topologies, we similarly generalize the concept of an isolated point, showing that this notion can be logically captured by $[i]$. As a bridge between the first part of the paper, which focuses on neighbourhood systems, and the second part, which is concerned solely with topological spaces, in §4 we consider the logic of discrete neighbourhood systems and discrete topologies. §5 extends the results of [7] by providing an axiomatic extension of the basic logic that is sound and complete with respect to the class of all frames in which the accessibility relation is a partial order. In §6, the correspondence between partial orders and Alexandrov T_0 topological spaces is leveraged to prove the paper's primary result, a topological characterization theorem for a logic of isolated points. We conclude in §7 and consider potential avenues of future research.

The current paper is a significant extension of [1]. That article (comprising roughly §2 through §4 of the current work) concludes by posing the open question of whether there are logics lying between $\mathbf{S^i}$ (the minimal $[i]$-logic with respect to relational semantics) and **SDisc** (the extension that was shown to capture the notion of isolation in discrete spaces) that characterize the isolated points operator in more general topological settings. The main purpose of this paper is to answer that question in the affirmative.

2 Syntax, proof system, and relational semantics

Take *Prop* to be a countably infinite set of propositional variables. The set *Form* of well-formed formulas of the language $\mathcal{L}^i$ is defined recursively as

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid [i]\alpha$$

for $p \in Prop$. The other Boolean operators are treated as abbreviations in the usual manner.

As mentioned in the introduction, the $[i]$ modality has been informally interpreted—specifically when introduced in [8]—as a *factive ignorance* operator: '$[i]\alpha$' was intended to mean something like, 'though "α" is true, its negation is believed (by an agent)'. As such, it is a stronger notion than being true but not believed. In this paper, however, we will not consider that interpretation. Instead, at a point x—in either a neighbourhood system or topological space—'$[i]\alpha$' will be understood as 'x is an isolated point of the set of points in which "α" is true'. This will be formally explicated below, in §3 and §4.

2.1 Relational semantics

Relational frames and models are defined as usual.

[2] Some authors use the term *neighbourhood system* to refer only to those families of neighbourhoods that give rise to a topological space. Our usage will be more liberal, allowing any set equipped with a neighbourhood function to qualify.

DEFINITION 2.1 (Relational Frame).
A *relational frame* is a pair $\langle X, R\rangle$ such that $X \neq \emptyset$ and $R \subseteq X^2$. A *relational model* is a pair $\langle F, V\rangle$, where F is a relational frame and $V : Prop \to \wp(X)$ is a valuation function.

Satisfaction of $\mathcal{L}^i$-formulas at points of a model is defined recursively, where the clause for the $[i]$ operator is:

$$M, x \models [i]\varphi \text{ iff } (M, x \models \varphi \text{ and } \forall y \neq x(xRy \text{ implies } M, y \not\models \varphi))$$

When no ambiguity can arise, we will omit the index pertaining to the model.

A formula is *valid* in a class of frames when it is satisfied at all points in all models based on frames in the class. A set of formulas is *satisfiable* in a class of frames when there is a point in a model based on a frame in the class at which all the elements of the set are satisfied. These definitions carry over, *mutatis mutandis*, to the settings of neighbourhood and topological frames.

2.2 Axiom system

The basic axiomatic proof system, $\mathbf{S^i}$, as defined in [7] (where it is referred to as $\mathbf{L^i}$), is as follows:

DEFINITION 2.2 (Proof System $\mathbf{S^i}$).

($Taut$) *All instances of propositional tautologies*

($A1$) $[i]\varphi \to \varphi$

($A2$) $([i]\varphi \wedge [i]\psi) \to [i](\varphi \vee \psi)$

(MP) *From* $\vdash \varphi$ *and* $\vdash \varphi \to \psi$ *infer* $\vdash \psi$

($R1$) *From* $\vdash \varphi \to \psi$ *infer* $\vdash \varphi \to ([i]\psi \to [i]\varphi)$

Our notions of *derivation*, *theorem*, and *consistency* are the usual ones.

PROPOSITION 2.3.
The following are all theorems of $\mathbf{S^i}$:

a. $([i]\varphi \wedge [i]\psi) \to [i](\varphi \wedge \psi)$
b. $[i](\varphi \vee \psi) \to ([i]\varphi \vee [i]\psi)$
c. $[i]\varphi \to [i][i]\varphi$

In addition, the rule allowing $\vdash [i]\varphi \leftrightarrow [i]\psi$ from $\vdash \varphi \leftrightarrow \psi$ is derivable.

THEOREM 2.4 ([7], Proposition 1.14 and Theorem 1.20).
$\mathbf{S^i}$ is sound and strongly complete with respect to the class of all relational frames.

3 Neighbourhood semantics for $\mathbf{S^i}$

As mentioned above, the ultimate goal of this paper is to demonstrate that $[i]$ can be interpreted as an isolated points operator. Recall the definitions of a topological space and an isolated point:

DEFINITION 3.1 (Topological Space).
Let X be a set and $\tau \subseteq \wp(X)$. τ is a *topology on* X when:

a. $\emptyset \in \tau$;
b. $X \in \tau$;

c. τ is closed under binary intersections;
d. τ is closed under arbitrary unions.

The elements of τ are referred to as *open sets*. $\mathcal{X} = \langle X, \tau \rangle$ is a *topological space* when τ is a topology on X.

DEFINITION 3.2 (Isolated Point).
Let $\mathcal{X} = \langle X, \tau \rangle$ be a topological space and $S \subseteq X$. x is an *isolated point* of S if there is an open neighbourhood U of x such that $U \cap S = \{x\}$.

Instead of beginning immediately with a purely topological semantics, we start by providing a more general semantic account in terms of neighbourhood systems (we follow [11] and [4], for instance, in the treatment of neighbourhood semantics).

DEFINITION 3.3 (Neighbourhood Frame).
A *neighbourhood frame* is a pair $\langle X, N \rangle$ such that $X \neq \emptyset$ and $N : X \rightarrow \wp(\wp(X))$. A *neighbourhood model* is a pair $\langle F, V \rangle$, where F is a neighbourhood frame and $V : Prop \rightarrow \wp(X)$ is a valuation function.

Generalizing the above definition of an isolated point to the context of neighbourhood systems, where less mathematical structure is insisted upon, one can say that x is an isolated point of S if there is a neighbourhood U of x, i.e. $U \in N(x)$, such that $U \cap S = \{x\}$. We can formalize this intuition in the definition of satisfaction for formulas in $\mathcal{L}^i$ with respect to neighbourhood models. Given a model M and a formula α, the *truth set* of α in M, denoted $[\![\alpha]\!]^M$, is defined via recursion:

$$
\begin{aligned}
[\![p]\!]^M &:= V(p) \\
[\![\neg\varphi]\!]^M &:= X \setminus [\![\varphi]\!]^M = ([\![\varphi]\!]^M)^c \\
[\![\varphi \wedge \psi]\!]^M &:= [\![\varphi]\!]^M \cap [\![\psi]\!]^M \\
[\![[i]\varphi]\!]^M &:= \{x : \exists U \in N(x) \text{ s.t. } U \cap [\![\varphi]\!]^M = \{x\}\}
\end{aligned}
$$

When no ambiguity can arise, the superscript will be omitted.

3.1 Semantic insensitivities

In the context of relational frames and the semantics given in §2, $\mathcal{L}^i$ is *reflexive-insensitive*. That is, the satisfaction of $\mathcal{L}^i$-formulas in a model $M = \langle X, R, V \rangle$ is not affected when arbitrary reflexive arrows are either added or removed. In the neighbourhood context, there are similar insensitivities.

In particular, because the definition of $[\![\cdot]\!]$ utilizes only sets of each $N(x)$ that contain x, the addition or removal of sets that do not contain x will be immaterial.

For a given neighbourhood frame $\langle X, N \rangle$, consider the set

$$S_x := \{Y \in \wp(X) : x \notin Y\}$$

for each $x \in X$.

Then, given a neighbourhood frame $F = \langle X, N \rangle$, construct the frames $F = \langle X, N^+ \rangle$ and $F = \langle X, N^- \rangle$, where N^+ and N^- are defined as follows for all $x \in X$:

$$N^+(x) := N(x) \cup S_x$$

$$N^-(x) := N(x) \setminus S_x$$

When M is a neighbourhood model, M^+ (M^-) is that model identical to M, but with N^+ (N^-) replacing N.

PROPOSITION 3.4.
Let M be a neighbourhood model. Then M, M^+, and M^- (as well as the intermediate models) are all pointwise equivalent. That is,

$$[\![\alpha]\!]^{M^-} = [\![\alpha]\!]^{M} = [\![\alpha]\!]^{M^+}$$

for all $\alpha \in Form$.

The proof is a straightforward induction. The case for the modalities simply makes use of the fact, remarked upon above, that the only differences between the frames on which M, M^+, and M^- are based concern the presence or absence of sets in each $N(x)$ that do not contain x. But such sets do not affect the evaluation of $[i]$-formulas.

However, in the current setting there are additional insensitivities that one can utilize.

DEFINITION 3.5 (Supplemented Neighbourhood System).
A neighbourhood frame is *supplemented* when its neighbourhood function is closed under supersets: for every x, if $Y \in N(x)$ and $Y \subseteq Z$, then $Z \in N(x)$.

Given a neighbourhood frame, $F = \langle X, N\rangle$, let $F^s = \langle X, N^s\rangle$ be the *supplementation* of F when, for all $x \in X$:

$$N^s(x) = \{Y \in \wp(X) : \exists U \subseteq Y \text{ s.t. } U \in N(x)\}$$

For a model $M = \langle F, V\rangle$, let $M^s = \langle F^s, V\rangle$.

REMARK 3.6.
Let M be a model and M^s its supplementation. Then it is not necessarily the case that

$$[\![\alpha]\!]^{M} = [\![\alpha]\!]^{M^s}$$

for $\alpha \in Form$.

The countermodels demonstrating this observation make use of supplementing some $N(x)$ containing at least one set from S_x and thereby adding sets to $N(x)$ not in S_x. (For instance, consider some state x such that $N(x) = \{\emptyset\}$. Then for no φ will it be the case that $x \in [\![[i]\varphi]\!]^M$. However, since $\{x\} \in N^s(x)$, $x \in [\![[i]\varphi]\!]^{M^s}$ for every φ such that $x \in [\![\varphi]\!]^M$. This is discussed further in §4, below.)

However, if no such sets are present in any $N(x)$ (for instance, as in neighbourhood filters in topological spaces), then supplementation will not affect satisfaction.

DEFINITION 3.7 (Anchored Neighbourhood System).
A neighbourhood function (and, hence, the resulting system) is *anchored* when, for every point $x \in X$,

$$\forall U \in N(x)(x \in U)$$

Note that we do not force $N(x) \neq \emptyset$ in order to be anchored.

PROPOSITION 3.8.
Let M be an anchored neighbourhood model. Then

$$[\![\alpha]\!]^M = [\![\alpha]\!]^{M^s}$$

for all $\alpha \in \mathit{Form}$.

PROOF. Induction on α. We omit all but the modal case.

If $x \in [\![[i]\varphi]\!]^M$, then there is some $U \in N(x)$ s.t. $U \cap [\![\varphi]\!]^M = \{x\}$. Since $U \in N^s(x)$ and $[\![\varphi]\!]^M = [\![\varphi]\!]^{M^s}$, by the induction hypothesis, $x \in [\![[i]\varphi]\!]^{M^s}$.

In the other direction, if $x \in [\![[i]\varphi]\!]^{M^s}$, there is a $U \in N^s(x)$ s.t. $U \cap [\![\varphi]\!]^{M^s} = \{x\}$. Thus, there must have been some $U_1 \in N(x)$ s.t. $U_1 \subseteq U$. But, since N is anchored, $x \in U_1$, and so $U_1 \cap [\![\varphi]\!]^{M^s} = \{x\}$. By the induction hypothesis, $[\![\varphi]\!]^{M^s} = [\![\varphi]\!]^M$, so $U_1 \cap [\![\varphi]\!]^M = \{x\}$. Hence, $x \in [\![[i]\varphi]\!]^M$. □

Propositions 3.8 and 3.4 together guarantee that, while a model might not be pointwise equivalent to its supplementation, it will be pointwise equivalent to *some* supplemented model.

COROLLARY 3.9.
Let M be any neighbourhood model. Then, for all $\alpha \in \mathit{Form}$,

$$[\![\alpha]\!]^M = [\![\alpha]\!]^{M^{-s}}$$

PROOF. This follows immediately from Propositions 3.4 and 3.8, along with the fact that M^- is anchored. □

3.2 Soundness and completeness

Using standard methods, characterization results for $\mathbf{S^i}$, with respect to the given neighbourhood semantics, are readily obtained. A logic is said to be *sound* with respect to a class of frames when all theorems of the logic are valid in the class. A logic is *complete* with respect to a class of frames when every consistent formula is satisfiable in the class. A logic is *strongly complete* with respect to a class when every consistent set of formulas is satisfiable in the class.

A neighbourhood frame $\langle X, N\rangle$ is said to be *closed under intersections* when, for every $x \in X$, if $U \in N(x)$ and $V \in N(x)$, then $U \cap V \in N(x)$.

THEOREM 3.10 (Soundness).
$\mathbf{S^i}$ is sound with respect to the class of neighbourhood frames that are closed under intersections.

PROOF. The proof is standard, and proceeds by showing that all axioms are valid and rules preserve validity. We include only the cases unique to $\mathbf{S^i}$.

$[i]\varphi \to \varphi$: Assume $x \in [\![[i]\varphi]\!]$. Then $\exists U \in N(x)$ s.t. $U \cap [\![\varphi]\!] = \{x\}$, so $x \in [\![\varphi]\!]$.

$([i]\varphi \wedge [i]\psi) \to [i](\varphi \vee \psi)$: Assume $x \in [\![[i]\varphi]\!]$ and $x \in [\![[i]\psi]\!]$. Then $\exists U_1 \in N(x)$ s.t. $U_1 \cap [\![\varphi]\!] = \{x\}$ and $\exists U_2 \in N(x)$ s.t. $U_2 \cap [\![\psi]\!] = \{x\}$. Since N is closed under intersections, $U_1 \cap U_2 \in N(x)$. But $(U_1 \cap U_2) \cap [\![\varphi \vee \psi]\!] = \{x\}$, so $x \in [\![[i](\varphi \vee \psi)]\!]$.

From $\vdash \varphi \to \psi$ infer $\vdash \varphi \to ([i]\psi \to [i]\varphi)$: Assume the validity of $\varphi \to \psi$. Then $[\![\varphi]\!] \subseteq [\![\psi]\!]$ in all models. Assume, further, $x \in [\![\varphi]\!]$. If $x \in [\![[i]\psi]\!]$, then $U \cap [\![\psi]\!] = \{x\}$, for some $U \in N(x)$. Since $[\![\varphi]\!] \subseteq [\![\psi]\!]$, $U \setminus \{x\} \subseteq X \setminus [\![\psi]\!] \subseteq X \setminus [\![\varphi]\!]$. Hence, $U \cap [\![\varphi]\!] = \{x\}$, so $x \in [\![[i]\varphi]\!]$. □

Our proof of completeness will proceed via a standard canonical model construction.

Let $\Sigma_{\mathbf{S^i}}$ be the set of all maximal $\mathbf{S^i}$-consistent sets and, for each $\alpha \in \mathit{Form}$,

$$|\alpha| := \{x \in \Sigma_{\mathbf{S^i}} : \alpha \in x\}$$

DEFINITION 3.11 (Neighbourhood Canonical Model for $\mathbf{S^i}$).
The neighbourhood canonical model for $\mathbf{S^i}$, $M^{\mathbf{S^i}} = \langle X^{\mathbf{S^i}}, N^{\mathbf{S^i}}, V^{\mathbf{S^i}} \rangle$, is as follows:

$X^{\mathbf{S^i}} := \Sigma_{\mathbf{S^i}}$
for each $x \in X^{\mathbf{S^i}}$, $N^{\mathbf{S^i}}(x) := \{|\neg\alpha| \cup \{x\} : [i]\alpha \in x\}$
for each $p \in Prop$, $V^{\mathbf{S^i}}(p) = |p|$

(The derivable rule mentioned in Proposition 2.3 ensures that N is well-defined).

Intuitively, the $N^{\mathbf{S^i}}$ function is constructed so as to add, for each $[i]\varphi \in x$, the largest set that is sufficient to ensure that x is in the truth set of $[i]\varphi$.

THEOREM 3.12 (Completeness).
$\mathbf{S^i}$ is strongly complete with respect to the class of neighbourhood frames that are anchored and closed under intersections.

PROOF. For all $\alpha \in Form$, a straightforward induction demonstrates that

$$[\![\alpha]\!]^{\mathbf{S^i}} = |\alpha|$$

The only non-trivial case is that of the modality. (In what follows, we omit all $\mathbf{S^i}$ superscipts.)

If $x \in |[i]\varphi|$, then, by definition, $|\neg\varphi| \cup \{x\} = (X \setminus |\varphi|) \cup \{x\} \in N(x)$. By the induction hypothesis, $|\varphi| = [\![\varphi]\!]$, so $(X \setminus [\![\varphi]\!]) \cup \{x\} \in N(x)$. Since $x \in |[i]\varphi \to \varphi|$ (from $A1$), $x \in |\varphi| = [\![\varphi]\!]$. Finally, $((X \setminus [\![\varphi]\!]) \cup \{x\}) \cap [\![\varphi]\!] = \{x\}$, so $x \in [\![[i]\varphi]\!]$.

If $x \in [\![[i]\varphi]\!]$, then there is some $U \in N(x)$ such that $U \cap [\![\varphi]\!] = \{x\}$ (hence, $x \in [\![\varphi]\!] = |\varphi|$, by the induction hypothesis). By construction, $U = |\neg\psi| \cup \{x\}$ for some ψ such that $x \in |[i]\psi|$ (and $x \in |\psi|$). But then $|\neg\psi| \subseteq |\neg\varphi|$, and so $|\varphi| \subseteq |\psi|$, meaning that $\vdash \varphi \to \psi$. Therefore, $\vdash \varphi \to ([i]\psi \to [i]\varphi)$ (from $R1$). Since $x \in |\varphi|$, $x \in |[i]\psi \to [i]\varphi|$. And, because $x \in |[i]\psi|$, $x \in |[i]\varphi|$.

Lastly, the model is both closed under intersections and anchored. Anchoring is by construction.

For closure under intersections, assume that $U_1, U_2 \in N(x)$. Then, $U_1 = |\neg\varphi_1| \cup \{x\}$ and $U_2 = |\neg\varphi_2| \cup \{x\}$ with $x \in |[i]\varphi_1|$ and $x \in |[i]\varphi_2|$. Since x is a maximal $\mathbf{S^i}$-consistent set, $x \in |[i]\varphi_1 \wedge [i]\varphi_2|$, and so $x \in |[i](\varphi_1 \vee \varphi_2)|$ (by $A2$). By construction, $|\neg(\varphi_1 \vee \varphi_2)| \cup \{x\} \in N(x)$. But $|\neg(\varphi_1 \vee \varphi_2)| = |\neg\varphi_1| \cap |\neg\varphi_2|$, and $U_1 \cap U_2 = (|\neg\varphi_1| \cap |\neg\varphi_2|) \cup \{x\}$, so $U_1 \cap U_2 \in N(x)$.

Hence, it easily follows that every $\mathbf{S^i}$-consistent set is satisfied at some point in $M^{\mathbf{S^i}}$, which is anchored and closed under intersections. □

COROLLARY 3.13.
$\mathbf{S^i}$ is strongly complete with respect to the class of neighbourhood frames that are anchored, closed under intersections, and supplemented.

PROOF. Since $M^{\mathbf{S^i}}$ is anchored, it is pointwise equivalent to its supplementation, from Proposition 3.8. Moreover, since supplementation interferes with neither anchoring nor closure under intersections, it follows that any $\mathbf{S^i}$-consistent set of formulas is satisfied on a model based on a frame that is anchored, closed under intersections, and supplemented. □

In addition, making use of the standard conversion between relational frames and augmented neighbourhood structures, a completeness theorem can also be obtained with respect to the class of all augmented neighbourhood frames.

DEFINITION 3.14 (Augmented Neighbourhood System).
A neighbourhood function (and, hence, the resulting system) is *augmented* when, for every point $x \in X$, $N(x)$ is supplemented and $\bigcap N(x) \in N(x)$.

LEMMA 3.15.
For every relational model, there is a pointwise equivalent neighbourhood model that is augmented.

PROOF. Let $M = \langle X, R, V\rangle$ be an arbitrary relational model. Define the function $N_R : X \to \wp(\wp(X))$ as

$$N_R(x) := \{U : R(x) \subseteq U\}$$

where $R(x) = \{y \in X : xRy\}$. Let $M^N = \langle X, N_R, V\rangle$. Note that N_R is augmented.

For all $\alpha \in \mathit{Form}$,

$$M, x \models \alpha \text{ iff } x \in [\![\alpha]\!]^{M^N}$$

This is, again, an induction on α and only the modal case will be discussed.

Assume $M, x \models [i]\varphi$. Then $M, x \models \varphi$ and $\forall y \neq x$, xRy implies $M, y \not\models \varphi$. Hence, $x \in [\![\varphi]\!]$ (from the induction hypothesis) and $R(x) \setminus \{x\} \subseteq X \setminus [\![\varphi]\!]$. Since N_R is augmented, $R(x) \cup \{x\} \in N_R$, and $(R(x) \cup \{x\}) \cap [\![\varphi]\!] = \{x\}$, so $x \in [\![[i]\varphi]\!]$.

Assume now that $M, x \not\models [i]\varphi$. Then either $M, x \not\models \varphi$ or, for some $y \neq x$ s.t. xRy, $M, y \models \varphi$.

If, $M, x \not\models \varphi$ then, by the induction hypothesis, $x \notin [\![\varphi]\!]$, and so $x \notin [\![[i]\varphi]\!]$.

Otherwise, assume that $M, x \models \varphi$ and $M, y \models \varphi$ for some $y \neq x$ s.t. xRy. Then $\{y\} \subseteq R(x)$ and $\{x, y\} \subseteq [\![\varphi]\!]$. Therefore, $\{y\} \subseteq U \cap [\![\varphi]\!]$ for all $U \in N_R(x)$, so $x \notin [\![[i]\varphi]\!]$. □

Clearly, if the original model was reflexive, then the resulting augmented model is anchored.

LEMMA 3.16.
For every augmented neighbourhood model, there exists a pointwise equivalent relational model.

PROOF. Let $M = \langle X, N, V\rangle$ be an arbitrary augmented neighbourhood model. Define the relational model $M^R = \langle X, R_N, V\rangle$ such that xR_Ny iff $y \in \bigcap N(x)$. Then, for all $\alpha \in \mathit{Form}$,

$$x \in [\![\alpha]\!]^M \text{ iff } M^R, x \models \alpha$$

Induction on α.

Assume $x \in [\![[i]\varphi]\!]$. Then there is a $U \in N(x)$ such that $U \cap [\![\varphi]\!] = \{x\}$. Hence, from the induction hypothesis, $M^R, x \models \varphi$. Moreover, $\bigcap N(x) \subseteq U$, so $\bigcap N(x) \setminus \{x\} \subseteq X \setminus [\![\varphi]\!]$. Therefore, for any $y \neq x$ such that $y \in \bigcap N(x)$, $y \in X \setminus [\![\varphi]\!]$, so $M^R, y \not\models \varphi$, by the induction hypothesis. Hence, $M^R, x \models [i]\varphi$.

In the other direction, if $M^R, x \models [i]\varphi$, then $M^R, x \models \varphi$ and $\forall y \neq x$, xR_Ny implies $M, y \not\models \varphi$. By the induction hypothesis, $x \in [\![\varphi]\!]$ and $\forall y \neq x$, if xR_Ny, then $y \notin [\![\varphi]\!]$. But xR_Ny iff $y \in \bigcap N(x)$. Hence, $\bigcap N(x) \setminus \{x\} \subseteq X \setminus [\![\varphi]\!]$. Let $U = \bigcap N(x) \cup \{x\}$. Then $U \in N(x)$, since $N(x)$ is supplemented. Moreover, $U \cap [\![\varphi]\!] = \{x\}$, so $x \in [\![[i]\varphi]\!]$. □

If the original neighbourhood system was anchored, then the resulting relational model is reflexive.

COROLLARY 3.17.
$\mathbf{S^i}$ is sound and strongly complete with respect to the class of all augmented neighbourhood frames and all anchored, augmented neighbourhood frames.

PROOF. For the anchored, augmented neighbourhood frames, strong completeness follows from taking the reflexive closure of the canonical model used in the proof of Theorem 2.4 (as defined in [7]) along with Lemma 3.15. □

4 Discrete neighbourhood systems

Before turning to the interpretation of the $[i]$ operator in topological spaces more generally, we consider the restricted class of discrete spaces. The supplementation of $\mathbf{S}^i$ by an additional, simple axiom is easily demonstrated sound and complete with respect to the class of relational frames in which the accessibility relation is a subset of id_X, the identity relation on X (where X is the set of points in the frame), as well as the class of discrete neighbourhood frames. The result for discrete neighbourhood frames is then easily transferred to discrete topological spaces.

Consider the following axiom schema:

$$\varphi \leftrightarrow [i]\varphi \qquad (Disc)$$

Call **SDisc** the system obtained by adding *Disc* to $\mathbf{S}^i$.

In the presence of *Disc*, no other modal axioms are necessary and neither is the rule $R1$.

PROPOSITION 4.1.
SDisc can be axiomatized by the following:

(*Taut*) *All instances of propositional tautologies*

(*Disc*) $\varphi \leftrightarrow [i]\varphi$

(*MP*) *From* $\vdash \varphi$ *and* $\vdash \varphi \to \psi$ *infer* $\vdash \psi$

PROPOSITION 4.2.
Disc is valid on a relational frame $\langle X, R\rangle$ if and only if, for each $x, y \in X$, if xRy, then $x = y$.

COROLLARY 4.3.
SDisc is sound with respect to the class of relational frames in which, for each x, y, if xRy, then $x = y$.

THEOREM 4.4.
SDisc is strongly complete with respect to the class of relational frames in which, for each x, y, if xRy, then $x = y$.

PROOF. This is easily seen by inspecting the relational canonical model for $\mathbf{S^i}$—as given in [7]—and observing that, with the addition of *Disc*, $[i]\top$ will be an element of each maximal consistent set. Since, by construction, $xR^{\mathbf{SDisc}}y$ if and only if $\{\neg\varphi : [i]\varphi \in x\} \subseteq y$, the accessibility relation in the canonical model for **SDisc** will be empty. □

In terms of neighbourhood systems, a characterization result for **SDisc** is also straightforward.

DEFINITION 4.5.
A neighbourhood system is *discrete* when $\{x\} \in N(x)$, for every $x \in X$.

THEOREM 4.6.
SDisc is sound and strongly complete with respect to the class of discrete neighbourhood systems. (Hence, due to the semantic insensitivities noted above, also with respect to anchored, discrete, supplemented neighbourhood systems).

PROOF. Soundness is immediate.

For completeness, one need only look at the canonical model construction from Definition 3.11, used in the proof of Theorem 3.12, and observe that, in the presence of *Disc*, $[i]\top \in x$, for every $x \in X^{\mathbf{SDisc}}$. Hence, $|\bot| \cup \{x\} = \{x\} \in N(x)$. □

4.1 Discrete topologies

We conclude the first part of the paper by turning our attention to topologies proper.

Recall the definition of a model based on a topological space.

DEFINITION 4.7 (Topo-Model).
A *topo-model* is a pair $M = \langle \mathcal{X}, V \rangle$ where $\mathcal{X} = \langle X, \tau \rangle$ is a topological space and $V : Prop \to \wp(X)$ is a valuation function.

Satisfaction at points in a topo-model is defined as usual, with the clause for $[i]$ defined in the obvious way (based on Definition 3.2) and resembling closely the one given for neighbourhood systems, but with reference to the topology τ rather than the neighbourhood function N:

$$[\![[i]\varphi]\!] := \{x : \exists U \in \tau \text{ s.t. } U \cap [\![\varphi]\!] = \{x\}\}$$

With the semantics so defined, $\mathbf{S^i}$ is sound with respect to the class of all topo-models (the proof is fundamentally the same as that of Theorem 3.10). Moreover, **SDisc** is sound with respect to the class of all discrete topological spaces, since all singletons are open.

The results above, concerning discrete neighbourhood systems, can be transferred over to the topological setting to render a completeness result for **SDisc** as well.

DEFINITION 4.8 (Neighbourhood Space).
Let X be a set and $N : X \to \wp(\wp(X))$. $\langle X, N \rangle$ is a *neighbourhood space* when, for each $x \in X$:

a. $N(x) \neq \emptyset$;
b. if $S \in N(x)$, then $x \in S$;
c. $N(x)$ is closed under supersets;
d. $N(x)$ is closed under intersections;
e. for each $S \in N(x)$, there is a $T \subseteq S$ such that $T \in N(x)$ and, for each $y \in T$, $S \in N(y)$.

Moreover, given a neighbourhood space $\langle X, N \rangle$, the pair $\langle X, \tau \rangle$ is a topological space when[3]

$$U \in \tau \text{ iff } \forall x \in U, U \in N(x)$$

THEOREM 4.9.
SDisc is complete with respect to the class of all discrete topological spaces.

PROOF. Consider $M^{\mathbf{SDisc}}$, the canonical neighbourhood model for **SDisc** (referred to in Theorem 4.6). The model is anchored and discrete. Let M^S be the supplementation of $M^{\mathbf{SDisc}}$. The frame of M^S is then a neighbourhood space. For readability, let the underlying set and neighbourhood function of M^S just be referred to as X and N (that is, $M^S = \langle X, N \rangle$). Consider the resulting (as in Definition 4.8) topological space $\mathcal{X} = \langle X, \tau \rangle$. Since, for each x, $\{x\} \in N(x)$, the topology is discrete. Let $M_{\mathcal{X}}$

[3] See, for instance, [10].

be the topo-model obtained by adding V, the valuation function from M^S, to $\mathcal{X}$. A straighforward induction proves that M^S and $M_{\mathcal{X}}$ are pointwise equivalent. □

5 Partial orders

For the duration of the paper, we will focus on the logic that is formed by adding the following axiom scheme to $\mathbf{S^i}$:

$$[i]\varphi \to [i](\varphi \vee ([i]\psi \wedge \neg[i](\psi \vee \varphi))) \qquad (Tr)$$

We will refer to the resulting logic as **STr**.[4]

The purpose of this section is to prove soundness and completeness results for **STr** with respect to the class of all relational frames in which the accessibility relation is a partial order. (Due to the semantic insensitivities of the logic, this will also provide soundness and completeness for strict partial orders.) Though our focus on this paper is on spatial interpretations of the $[i]$ operator, the relationship between partial orders and topologies will provide an avenue to our desired results.

LEMMA 5.1.
Tr is valid on the class of relational frames in which R is antisymmetric and transitive.

PROOF. Assume not. Then there is some point x in a model based on an antisymmetric, transitive frame such that

$$x \not\models Tr$$

Then $x \models [i]\varphi$ and, also,

$$\text{for every } y \neq x, \text{ if } xRy, \text{ then } y \not\models \varphi \qquad (1)$$

In addition, $x \not\models [i](\varphi \vee ([i]\psi \wedge \neg[i](\psi \vee \varphi)))$. Since $x \models \varphi$, it must be that $x \models (\varphi \vee ([i]\psi \wedge \neg[i](\psi \vee \varphi)))$ and so there must be some $y \neq x$ such that xRy and $y \models (\varphi \vee ([i]\psi \wedge \neg[i](\psi \vee \varphi)))$. From 1, $y \not\models \varphi$, so $y \models [i]\psi \wedge \neg[i](\psi \vee \varphi)$. Hence, $y \models [i]\psi$, which gives both $y \models \psi$ and also

$$\text{for every } z \neq y, \text{ if } yRz, \text{ then } z \not\models \psi \qquad (2)$$

Because $y \models \psi$, $y \models \psi \vee \varphi$, and so, from $y \models \neg[i](\psi \vee \varphi)$, it follows that there is a $z \neq y$ such that yRz and $z \models \psi \vee \varphi$.

From 2, $z \not\models \psi$. But from 1, the assumed transitivity of the model, and the fact that $z \neq x$ (from antisymmetry), it also must be that $z \not\models \varphi$. Hence, $z \not\models \psi \vee \varphi$, a contradiction. □

COROLLARY 5.2.
STr is sound with respect to the class of frames in which R is antisymmetric and transitive. Hence, it is also sound with respect to the class of partial orders and the class of strict partial orders.

5.1 Completeness

We will prove the completeness of **STr** (with respect to partial orders) via a canonical model construction. Unfortunately, as explained below, the canonical model for $\mathbf{S}^i$ defined in [7] is not

[4]This is a simplified version of the translation—based on the discussion in the second half of [7]—of the axiom scheme *A*4 that is introduced in [6] in the context of logics for the 'being wrong' operator of [14]. In [6], it is shown that the logic resulting from the addition of *A*4—which is $W\psi \wedge W(\varphi \wedge \psi) \to W((W\chi \to W(\varphi \wedge \chi)) \wedge \psi)$—to the minimal logic of being wrong is sound and complete with respect to transitive frames.

easily adapted to facilitate a proof of the desired result for **STr**. Because of this, we first define a different, new canonical model for $\mathbf{S}^i$ that is then more easily modified for our purposes.[5] We include this brief detour—rather than proceeding directly to the modified constructions given in Definitions 5.8 and 5.11, pertinent only to **STr**—in the interest of both generality and readability.

As in [7], define the function $f : \wp(Form) \rightarrow \wp(Form)$ as

$$f(x) := \{\neg\varphi : [i]\varphi \in x\}$$

$f(x)$ needn't be consistent. (If $[i]\top \in x$, for instance, it obviously cannot be.)

Let $\Sigma := \{x : x \text{ is maximal} \mathbf{S}^{\mathbf{i}}\text{-consistent}\}$.

When proving completeness for extensions of $\mathbf{S}^{\mathbf{i}}$, particularly **STr**, Σ will be tacitly understood to be the set of maximal sets consistent with respect to the extension.

PROPOSITION 5.3 ([7], Proposition 1.15).
Let $x \in \Sigma$. If $[i]\varphi \notin x$ and $\varphi \in x$, then $\{\varphi\} \cup f(x)$ is consistent. Moreover, if $f(x)$ is non-empty, then $f(x) \cup \{\varphi\} \subseteq y$ for some $y \in \Sigma$ such that $x \neq y$.

Let $L \subseteq \Sigma$ be defined as

$$L := \{x \in \Sigma : f(x) = \emptyset\}$$

L contains the sets that contain no formulas of the form $[i]\varphi$. (Intuitively, and with an eye towards the topological semantics, they can be thought of as those points that are limit points for *all* $[\![\varphi]\!]$.)

In contrast, define $I \subseteq \Sigma$ as

$$I := \{x \in \Sigma : f(x) \neq \emptyset\}$$

Clearly, then, $\Sigma = L \cup I$ and $L \cap I = \emptyset$.

In the canonical model in [7], for every $x, y \in \Sigma$, the accessibility relation was defined so that xRy if and only if $f(x) \subseteq y$. Hence, for $x \in L$, the accessibility relation was universal: if $f(x) = \emptyset$, then xRy for all y. This approach had the advantage of a homogeneous treatment for all elements of Σ. However, adopting such a definition when working with **STr**, our actual focus, impedes the construction of a transitive model. Instead, we specify R in a piecewise manner, based on whether a point is in L or I.[6] As a result, instead of making an $x \in L$ have access to every point, here we 'clone' it, and make x and its clone see exactly each other, forming a two point, irreflexive cycle (and, hence, forcing all $[i]$-formulas to be false at those two points).

This modification alone does not suffice immediately to render the completeness result with respect to partial orders, but will facilitate the necessary subsequent alterations.

DEFINITION 5.4 (Canonical Model for Extensions of $\mathbf{S}^{\mathbf{i}}$).
The canonical model for extensions of $\mathbf{S}^{\mathbf{i}}$, $M^c = \langle X^c, R^c, V^c\rangle$ is as follows:

$X^c = (I \times \mathbf{1}) \cup (L \times \mathbf{2})$
$V^c(p) = \{\langle x, i\rangle : p \in x\}$
if $x \in L$, then $\langle x, i\rangle R^c \langle y, j\rangle$ iff $x = y$ and $j = 1 - i$
if $x \in I$, then $\langle x, i\rangle R^c \langle y, j\rangle$ iff $f(x) \subseteq y$

(For notational convenience, we will write x_i in place of $\langle x, i\rangle$.)

[5] Our canonical model construction differs from the one in [7] only with respect to how the accessibility relation is defined.
[6] Separating the cases in this way is not novel, and a similar approach is taken in the canonical model construction in [14], where the focus was on logics of being wrong.

When $f(x)$ is not consistent (meaning $x \in I$), there is obviously no y such that $f(x) \subseteq y$ and so there is no y_j for which $x_0 R^c y_j$.

For each $x \in I$ there is an x_0 but not an x_1 and the definition of R^c will essentially be as before. On the other hand, for each $x \in L$, there is an x_0 and an x_1 that each has access to the other and nothing else. Of course, if $y \in L$ and $x \in I$ such that $f(x) \subseteq y$, then both $x_0 R^c y_0$ and $x_0 R^c y_1$.

LEMMA 5.5 (Truth Lemma).
For all $\alpha \in \mathit{Form}$ and all $x \in \Sigma$ and $i \in \mathbf{2}$:

$$M^c, x_i \models \alpha \text{ iff } \alpha \in x$$

PROOF. The proof is by induction on α. We only demonstrate the modal case.

Assume $[i]\varphi \in x$. Then $\varphi \in x$ and $M^c, x_i \models \varphi$, by the induction hypothesis. Furthermore, $\neg\varphi \in f(x)$ so $\neg\varphi \in y$ for every y such that $x_i R^c y_j$. Hence, each y_j is such that $M^c, y_j \models \neg\varphi$ (induction hypothesis), so $M^c, x_i \models [i]\varphi$.

Assume now that $[i]\varphi \notin x$. Either $\varphi \notin x$ or $\varphi \in x$.

In the first case, $\varphi \notin x$ and $M^c, x_i \not\models \varphi$, by the induction hypothesis, so $M^c, x_i \not\models [i]\varphi$.

In the second case, from Proposition 5.3, $\{\varphi\} \cup f(x)$ is consistent. Moreover, if $f(x)$ is non-empty, then $f(x) \cup \{\varphi\} \subseteq y$ for some $y \in \Sigma$ such that $x \neq y$. Hence, from the induction hypothesis, $M^c, y_i \models \varphi$ and $x_i R^c y_i$, thus $M^c, x_i \not\models [i]\varphi$.

If $f(x)$ is empty, then $x_i R^c x_j$, where $j = 1 - i$. But since $\varphi \in x_i$, $\varphi \in x_j$. So, $M^c, x_j \not\models \varphi$, from the induction hypothesis, and $M^c, x_i \not\models [i]\varphi$. □

COROLLARY 5.6.
$\mathbf{S^i}$ (as well as all extensions, including **STr**, in particular) is strongly complete with respect to the class of all frames.

This is, of course, not a new result. But the purpose of the new canonical construction is, ultimately, to facilitate a proof of completeness for **STr** with respect to partially ordered frames. However, in light of the following observation, some adjustments are still needed.

PROPOSITION 5.7.
As defined, the canonical frame for **STr** is irreflexive and not transitive.

PROOF. Irreflexivity: If $x \in I$, then $f(x) \not\subseteq x$. If $x \in L$, then, by definition of R^c, it is not the case that $x_i R^c x_i$.

Transitivity: For every $x \in L$, $x_i R^c x_j$ and $x_j R^c x_i$ but not $x_i R^c x_i$, by construction. □

Given the semantic insensitivities of $[i]$-logics, the lack of reflexivity can be immediately remedied. Moreover, adding all reflexive arrows will also resolve the failures of transitivity amongst the points in L. However, the resulting frame will still fail to be anti-symmetric.[7] The following construction resolves this difficulty.

Definition 5.8.
Let M^c be a canonical model for an extension of $\mathbf{S^i}$, as constructed above. Construct $M^t = \langle X^t, R^t, V^t \rangle$ as follows:

[7]It is easy to show that taking the reflexive closure of R^c produces a preorder, which, in turn, gives a completeness result for **STr** with respect to the class of preorders and, subsequently, when we turn to topo-models, facilitates a proof of completeness with respect to the class of all Alexandrov spaces. However, those results do not match the soundness results of Corollaries 5.2 and 6.14, therefore failing to be adequate for characterization theorems. Obviously, however, completeness for preorders as well as for Alexandrov spaces are immediately implied by the completeness results for partial orders and Alexandrov T_0 spaces that are proved below.

$X^t = (I \times \mathbf{1}) \cup (L \times \omega)$
$V^t(p) = \{\langle x, i\rangle : p \in x\}$
if $x \in I$, then $x_0 R^t y_i$ iff $f(x) \subseteq y$ (i.e. iff $x_0 R^c y_0$)
if $x \in L$, then $x_i R^t y_j$ iff $x = y$ and $i < j$

Intuitively, M^t is constructed from M^c by removing the symmetric clone pairs based on points from L and replacing them with infinite chains of modally equivalent (with respect to $\mathcal{L}^i$) points.

PROPOSITION 5.9.
If $x \in L$, then, for all $\alpha \in Form$, $M^c, x_0 \models \alpha$ iff $M^t, x_i \models \alpha$ for all $i \in \omega$. If $x \in I$, $M^c, x_0 \models \alpha$ iff $M^t, x_0 \models \alpha$ for all α.

PROOF. For $x \in L$ the result follows from a straightforward induction where the base case is a consequence of the definitions of V^t and V^c and the modal case follows from the fact that in both models, it is never the case that $x \models [i]\varphi$ for any φ.

The statement for $x \in I$ then follows by a similarly straightforward induction. □

The immediately preceding definition and proposition apply for all extensions of $\mathbf{S^i}$. However, our focus will now rest now solely on **STr**. Subsequently, the base logic is assumed to be **STr**, and the points in the models M^c and M^t are indexed **STr**-consistent sets.

LEMMA 5.10.
The model M^t, as defined, is transitive and asymmetric (hence, irreflexive), and so a strict partial order.

PROOF. Let $x, y, z \in X^t$ such that $x_i R^t y_j R^t z_k$. We consider three cases.

In the first, if $x \in L$, then $x = y$ and $i < j$. But then $y \in L$ and so $y = z$ with $j < k$. Hence $x = z$ with $i < k$ so $x_i R^t z_k$.

In the second, let $x \in I$ but $y \in L$. Then $f(x) \subseteq y$ and $y = z$. Hence $f(x) \subseteq z$ and so $x_i R z_k$.

In the final case, assume that $x, y \in I$. In that case, neither $f(x)$ nor $f(y)$ are empty.

Let $\neg\varphi$ be an arbitrary element of $f(x)$. Then $[i]\varphi \in x$.

Since $f(y)$ is not empty, there is at least one ψ such that $[i]\psi \in y$.

From *Tr*,

$$[i]\varphi \to [i](\varphi \vee ([i]\psi \wedge \neg[i](\psi \vee \varphi))) \in x$$

Thus,

$$[i](\varphi \vee ([i]\psi \wedge \neg[i](\psi \vee \varphi))) \in x$$

Hence

$$\neg(\varphi \vee ([i]\psi \wedge \neg[i](\psi \vee \varphi))) \in f(x) \subseteq y$$

Therefore, $\neg\varphi \in y$ and $\neg([i]\psi \wedge \neg[i](\psi \vee \varphi)) \in y$.

From the maximality of y, it must be that either $\neg[i]\psi \in y$ or $[i](\psi \vee \varphi) \in y$. Therefore, since $[i]\psi \in y$ by assumption, and y is consistent, $[i](\psi \vee \varphi) \in y$. But then $\neg(\psi \vee \varphi) \in f(y) \subseteq z$ and so $\neg\varphi \in z$, by maximality and consistency.

Lastly, R^t is still irreflexive (so asymmetric as well), and so it is a strict partial order. □

Definition 5.11.
Define $M^{\leq} = \langle X^{\leq}, R^{\leq}, V^{\leq}\rangle$ such that $X^{\leq} = X^t$, $V^{\leq} = V^t$, and $R^{\leq}$ is the reflexive closure of R^t.

THEOREM 5.12.
STr is strongly complete with respect to the class of all strict partial orders (and all superclasses). In addition, it is strongly complete with respect to the class of all partial orders (and all superclasses).

PROOF. The construction of the canonical model along with Lemma 5.5 ensure that **STr** is strongly complete with respect to any class of frames containing the frame of the canonical model, M^c. Proposition 5.9 along with Lemma 5.10 show that any set of $\mathcal{L}^i$-formulas satisfied in M^c is also satisfied in M^t, the frame of which is a strict partial order.

Lastly, M^t and $M^{\leq}$ are modally equivalent (in $\mathcal{L}^i$) and the frame of $M^{\leq}$ is a partial order. □

6 Topological isolation

We now shift our attention solely to topological spaces. As stated in §4, $[i]$ is interpreted in topo-models as the isolated points operator:

$$[\![[i]\varphi]\!] := \{x : \exists U \in \tau \text{ s.t. } U \cap [\![\varphi]\!] = \{x\}\}$$

The primary purpose of this section is to prove a topological characterization theorem for **STr**. In particular, we prove that **STr** is sound and strongly complete with respect to the class of all topo-models based on Alexandrov T_0 spaces. Our proofs build on the results of the previous section by leveraging the well-known correspondence between Alexandrov T_0 spaces and partial orders.[8]

We briefly review some definitions and results concerning topological spaces and partial orders (mostly without proof) before proving the soundness and completeness theorems.

DEFINITION 6.1 (Kolmogorov Space).
A *Kolmogorov* space, or T_0 space, is a topological space in which, for every pair of distinct points, one of the points has a neighbourhood not containing the other. That is, every pair of distinct points are *topologically distinguishable*.

DEFINITION 6.2 (Alexandrov Space).
An *Alexandrov* space is a topological space in which the topology is closed under arbitrary intersections.

Definition 6.3.
Let $\langle X, \tau \rangle$ be an Alexandrov space and $x \in X$.

$$O_x := \bigcap \{U \in \tau : x \in U\}$$

O_x is the *smallest* (or *minimal*) neighbourhood of x.

DEFINITION 6.4 (Specialization Order).
Let $\langle X, \tau \rangle$ be a topological space. The relation $\sqsubseteq_\tau$, defined by

$$x \sqsubseteq_\tau y \text{ iff } x \in \overline{\{y\}}$$

is called the *specialization order* where $\overline{S}$ denotes the topological closure of set S.

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[8] See, for instance, [2].

Because $x \in \overline{Y}$ if and only if $(U \cap Y) \neq \emptyset$ for every open neighbourhood U of x, it is equivalent to define $\sqsubseteq_\tau$ as

$$x \sqsubseteq_\tau y \text{ iff } \forall U \in \tau (\text{if } x \in U, \text{ then } y \in U)$$

We make use of the following standard results and definitions.

PROPOSITION 6.5.
Let $\langle X, \tau \rangle$ be a topological space.

1. $\sqsubseteq_\tau$ is a preorder;
2. $\langle X, \tau \rangle$ is a T_0 space if and only if $\sqsubseteq_\tau$ is a partial order.

PROPOSITION 6.6.
Let $\langle X, \tau \rangle$ be an Alexandrov space. Then

1. $x \sqsubseteq_\tau y$ iff $y \in O_x$;
2. $x \sqsubseteq_\tau y$ iff $O_y \subseteq O_x$;
3. $x \in X$ is an isolated point of $S \subseteq X$ iff $O_x \cap S = \{x\}$.

DEFINITION 6.7 (Upset).
Let X be a set and $R \subseteq X^2$. $S \subseteq X$ is said to be an *upset* of $\langle X, R \rangle$ if $x \in S$ and xRy implies that $y \in S$. $S \subseteq X$ is a *downset* if $x \in S$ and yRx implies that $y \in S$. For $x \in X$, $U_x := \{y : xRy\}$ is said to be a *cone* and x is called the *root* of U_x.[9]

If R is transitive on X, then each U_x is an upset of $\langle X, R \rangle$. If it is reflexive, then $x \in U_x$.

PROPOSITION 6.8.
Let X be a set with $R \subseteq X^2$ such that R is a preorder. Define $\tau_R \subseteq \wp(X)$ such that $S \in \tau_R$ iff S is an upset of $\langle X, R \rangle$. Then τ_R is an Alexandrov topology on X. Moreover, the downsets of $\langle X, R \rangle$ are closed.

If R is a partial order, then τ_R is also a T_0 topology.

(Note that in τ_R, $U_x = O_x$).

As mentioned above, our proofs in the subsequent sections rely on the 1-1 correspondence between partial orders and Alexandrov T_0 spaces. More generally, if $\langle X, \leq \rangle$ is a preorder, then the specialization order $\sqsubseteq_{\tau_\leq}$ is equal to $\leq$; if $\langle X, \tau \rangle$ is an Alexandrov space, then the topology $\tau_{\sqsubseteq_\tau}$ is equal to τ.

Our soundness and completeness theorems then follow, more or less straightforwardly, from the observation that the formulas of $\mathcal{L}^i$ are invariant when moving between topo-models and their corresponding relational models, where the relation is the topology's specialization order.

LEMMA 6.9.
Let $M = \langle X, \tau, V \rangle$ be a topo-model where $\langle X, \tau \rangle$ is an Alexandrov space and $M^{\sqsubseteq_\tau} = \langle X, \sqsubseteq_\tau, V \rangle$ be the relational model where $\sqsubseteq_\tau$ is the specialization order on X. Then, for all $\alpha \in \mathit{Form}$ and $x \in X$,

$$M^{\sqsubseteq_\tau}, x \models \alpha \text{ iff } x \in [\![\alpha]\!]^M$$

PROOF. The proof is by induction on α.

[9] $R(x)$, from Lemma 3.15, above, is what we are here denoting U_x.

If $M^{\sqsubseteq_\tau}, x \not\models [i]\varphi$, then either $M^{\sqsubseteq_\tau}, x \not\models \varphi$ or else there is a y such that $y \neq x$, $x \sqsubseteq_\tau y$, and $M^{\sqsubseteq_\tau}, y \models \varphi$. In the first case, by the induction hypothesis, $x \notin [\![\varphi]\!]^M$ and so $x \notin [\![[i]\varphi]\!]^M$.

In the second case, by the induction hypothesis, $y \in [\![\varphi]\!]^M$. Since $x \sqsubseteq_\tau y$, $y \in O_x$ and so $\{x, y\} \subseteq [\![\varphi]\!]^M \cap O_x$. Thus x is not an isolated point of $[\![\varphi]\!]^M$ and so $x \notin [\![[i]\varphi]\!]^M$.

In the other direction, if $x \notin [\![[i]\varphi]\!]^M$, then for all $U \in \tau$ such that $x \in U$, $U \cap [\![\varphi]\!]^M \neq \{x\}$. In particular, $[\![\varphi]\!]^M \cap O_x \neq \{x\}$.

If $[\![\varphi]\!]^M \cap O_x = \emptyset$, then $x \notin [\![\varphi]\!]$ and so, by the induction hypothesis, $M^{\sqsubseteq_\tau}, x \not\models \varphi$ and so $M^{\sqsubseteq_\tau}, x \not\models [i]\varphi$.

Otherwise, there is some $y \neq x$ such that $\{y, x\} \subseteq [\![\varphi]\!]^M \cap O_x$. Hence $y \in O_x$ and so $x \sqsubseteq_\tau y$ with $M^{\sqsubseteq_\tau}, y \models \varphi$ (by induction hypothesis). Hence, $M^{\sqsubseteq_\tau}, x \not\models [i]\varphi$. □

6.1 Soundness

We begin by proving a soundness theorem that does not rely on the topo-models being built on Alexandrov spaces.

DEFINITION 6.10 (Fréchet Space).
A *Fréchet* space, or T_1 space, is a topological space in which, for every pair of distinct points, each point has a neighbourhood not containing the other. That is, every pair of points is *(topologically) separated*.

LEMMA 6.11.
Tr is valid on the class of all T_1 topo-models.

PROOF. Assume not. Then there is a T_1 model containing a point x such that

$$x \notin [\![Tr]\!]$$

Then, $x \in [\![[i]\varphi]\!]$. That is, x is an isolated point of $[\![\varphi]\!]$, and so there is some $U \in \tau$ such that $U \cap [\![\varphi]\!] = \{x\}$.

Also, $x \notin [\![[i](\varphi \vee ([i]\psi \wedge \neg[i](\psi \vee \varphi)))]\!]$ (i.e. x is not an isolated point of $[\![\varphi \vee ([i]\psi \wedge \neg[i](\psi \vee \varphi))]\!]$). It follows that there is some $y \neq x$ such that: $y \in U$, $y \notin [\![\varphi]\!]$, $y \in [\![[i]\psi]\!]$, and $y \notin [\![[i](\psi \vee \varphi)]\!]$.

From $y \in [\![[i]\psi]\!]$ it follows that, for some $O \in \tau$, $O \cap [\![\psi]\!] = \{y\}$.

From $y \notin [\![[i](\psi \vee \varphi)]\!]$ it follows that

$$\forall O' \in \tau, O' \cap [\![(\psi \vee \varphi)]\!] \neq \{y\} \tag{3}$$

Since x and y are separable, y has an open neighbourhood that does not contain x. Call this neighbourhood Y.

Consider now $(Y \cap O \cap U) \in \tau$. Clearly, $y \in (Y \cap O \cap U)$. Since $y \in [\![\psi]\!] \subseteq [\![(\psi \vee \varphi)]\!]$, $(Y \cap O \cap U) \cap [\![(\psi \vee \varphi)]\!] \neq \emptyset$. From 3, it then follows that there is some $z \neq y$ such that $z \in (Y \cap O \cap U) \cap [\![(\psi \vee \varphi)]\!]$.

Because $O \cap [\![\psi]\!] = \{y\}$, $z \notin [\![\psi]\!]$. Hence, it must be the case that $z \in [\![\varphi]\!]$. But then $z \in U \cap [\![\varphi]\!]$. Hence, $z = x$. But then $x \in Y$, a contradiction. □

COROLLARY 6.12.
STr is sound with respect to the class of all T_1 topo-models (and all subclasses).

However, as already suggested, if attention is restricted to Alexandrov topologies, the T_1 constraint can be relaxed.

LEMMA 6.13.
Tr is valid on the class of all topo-models where the underlying topology is both Alexandrov and T_0.

PROOF. If *Tr* is not valid on the class of all models based on Alexandrov T_0 spaces, then there is some such model $M = \langle X, \tau, V \rangle$ and $x \in X$ such that $x \notin [\![Tr]\!]^M$. But then, by Lemma 6.9, $M^{\sqsubseteq_\tau}, x \not\models Tr$, which contradicts the fact that *Tr* is valid on the class of all relational models in which the accessibility relation is a partial order (Lemma 5.1). □

COROLLARY 6.14.
STr is sound with respect to the class of all Alexandrov T_0 topo-models (and all subclasses).

6.2 Completeness

We now conclude with our primary topological completeness and characterization results.

THEOREM 6.15.
STr is strongly complete with respect to the class of all Alexandrov T_0 topologies (and all superclasses).

PROOF. Consider the model $M^{\leq} = \langle X^{\leq}, R^{\leq}, V^{\leq} \rangle$, from Definition 5.11. From Lemma 6.9, $M^{\leq}$ is pointwise equivalent to the Alexandrov T_0 topo-model M for which $\leq$ is the specialization order. Hence, since all **STr**-consistent sets are satisfied in $M^{\leq}$, they are satisfied in M. □

Finally, then, combined with Corollary 6.14, we obtain a topological characterization theorem of **STr**.

THEOREM 6.16.
STr is characterized by the class of all topo-models based on Alexandrov T_0 topological spaces.

Hence, **STr** is the logic of isolated points in Alexandrov T_0 spaces.

7 Conclusion and future work

We have argued that there is a plausible interpretation of $[i]$ as an isolated points operator in both a variety of neighbourhood systems—including those that correspond to discrete topologies—as well as in the topological setting of Alexandrov T_0 spaces. We consider this work merely a first step en route to a more robust logical understanding of spatial isolation. It remains unresolved whether other, or more general, topological characterization results are forthcoming for extensions of $\mathbf{S^i}$. Given the results pertaining to neighbourhood systems above—particular the bookending characterization results in Theorems 3.12 and 4.6—it would seem that the answers to at least some these questions hinge upon the expressivity of the $[i]$ operator in neighbourhood and topological settings. A more thorough investigation of these issues is left for subsequent work.

Perhaps an even more attractive avenue is that devoted to understanding the topological character of the semantic insensitivities of $\mathcal{L}^i$ along with those languages of other non-normal modal operators that exhibit similar behaviours. This too, we hope to address in future projects.

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References

[1] C. Başkent, D. Gilbert and G. Venturi. A logic of isolation. In *Logic, Language, Information, and Computation*, G. Metcalfe, T. Studer and R. de Queiroz, eds. WoLLIC 2024. Lecture Notes in Computer Science, vol. **14672**, pp. 36–46. Springer, Cham, 2024.

[2] J. van Benthem and G. Bezhanishvili. Modal logics of space. In *Handbook of Spatial Logics*, M. Aiello, I. Pratt-Hartmann and J. van Benthem, eds, pp. 217–298. Springer, 2007.

[3] G. Bezhanishvili, L. Esakia and D. Gabelaia. Some results on modal axiomatization and definability for topological spaces. *Studia Logica*, **81**, 325–355, 2005.

[4] B. Chellas. *Modal Logic: An Introduction*. Cambridge University Press, 1980.

[5] L. Esakia. Intuitionistic logic and modality via topology. *Annals of Pure and Applied Logic*, **127**, 155–170, 2004.

[6] J. Fan. *Logics of false belief and radical ignorance*. arXiv, 2024, arXiv:2404.07442.

[7] D. Gilbert, E. Kubyshkina, M. Petrolo and G. Venturi. Logics of ignorance and being wrong. *Logic Journal of the IGPL*, **30**, 870–885, 2021.

[8] E. Kubyshkina and M. Petrolo. A logic for factive ignorance. *Synthese*, **198**, 5917–5928, 2021.

[9] J. C. C. McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, **45**, 141–191, 1944.

[10] B. Mendelson. *Introduction to Topology*, 3d edn., Allyn and Bacon, 1974.

[11] E. Pacuit. *Neighborhood Semantics for Modal Logic*. Springer, 2017.

[12] V. Shehtman.. *Derived Sets in Euclidean Spaces and Modal Logic* Technical report,. The University of Amsterdam, 1990.

[13] C. Steinsvold. A note on logics of ignorance and borders. *Notre Dame Journal of Formal Logic*, **49**, 385–392, 2008.

[14] C. Steinsvold. Being wrong: Logics for false belief. *Notre Dame Journal of Formal Logic*, **52**, 245–253, 2011.

[15] T.-C. Tang. Algebraic postulates and a geometric interpretation for the Lewis calculus of strict implication. *Bulletin (new series) of the American Mathematical Society*, **44**, 737–744, 1938.

[16] A. Tarski. Der Aussagenkalkül und die Topologie. *Fundamenta Mathematicae*, **31**, 103–134, 1938.

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